General Lyapunov-Based Iterative Algorithm for Linear Quadratic Regulator Problem of Stochastic Systems with Markovian Jump

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Abstract

This paper investigates the linear quadratic regulator(LQR) problem of linear stochastic systems with Markovian jump. Firstly, two iterative algorithms are proposed for solving the corresponding coupled algebraic Riccati equa- tions (CAREs) based on the general-type Lyapunov equation derived from linear stochastic systems. It is verified that the second algorithm adding an adjustable factor converges faster than the first one without it. Secondly, a monotonic convergence theorem is established for the proposed iterative algorithms under certain initial conditions. In the end, a numerical example is given to verify the efficiency of the proposed algorithms.

figures/example1/example1-eps-converted-to.pdf figures/example2/example2-eps-converted-to.pdf

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Abstract

This paper investigates the linear quadratic regulator(LQR) problem of linear stochastic systems with Markovian jump. Firstly, two iterative algorithms are proposed for solving the corresponding coupled algebraic Riccati equations (CAREs) based on the general-type Lyapunov equation, which were derived from linear stochastic systems. It has been verified that the proposed new algorithm not only provides a solution for the LQR problem of stochastic systems, but also improves the convergence speed compared with the existing ones. Secondly, a monotonic convergence theorem is established for the proposed iterative algorithms under certain initial conditions. In the end, a numerical example is given to verify the efficiency of the proposed algorithms.

Keywords: Stochastic systems, General Lyapunov equation, Iterative Algorithms, Markovian Jump

1. Introduction

The problem of linear regulator (LQR) problem has attracted a lot of attention in the field of control theory and applied mathematics, and scholars in the field of control theory have published a lot of relevant literature on this type of problem, see [1, 2, 3, 4]. The main idea of LQR problem is to find an

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optimal control law such that the system achieves optimal performance with low cost, which is simple to implement and easy to simulate.

As a heated research topic, systems driven by continuous-time Markovian chains have drawn a lot of attention. This type of hybrid system has been used to model many practical systems that may experience sudden changes in structure and parameters caused by component failures or maintenance, changing subsystems in interconnection, and sudden environmental interference[5]. Models with Markovian jump are usually applied to describe economic systems, electrical power systems, robot manipulator systems, and communication systems[6, 7, 8], therefore systems with Markovian jump are widely considered and studied.

The LQR problem of linear systems with Markovian jump(JLQR) is discussed see [9, 10], and the optimal solution of this kind of problem can be obtained by solving the corresponding coupled algebraic Riccati equations(CAREs), which are complicated due to their nonlinear and coupled terms. Many algorithms for solving the solution of CAREs have been presented over the last couple of decades. For example, [11] put forward an iterative calculation algorithm based on Newton's method, which synthesized all modes of coefficient matrices into a single matrix. Theoretically, this algorithm is applicable for solving matrix equations of any dimension, but it may cause a dimensional disaster due to the high dimension of the corresponding matrix. [12] developed a Lyapunov-based iterative algorithm to decouple the CAREs by constructing standard Lyapunov iterative equations, which greatly simplifies the calculation. For better convergence performance, [13] generalized the work of [12] to a new one combined with an initial value selection method. By choosing an appropriate initial value, the solution sequence obtained by [13] is monotonic. Without loss of generality, an adjustable factor can be added to adjust the updated information to accelerate the convergence speed to a greater extent, see [14, 15, 16].

Later, the investigation on JLQR problem is extended to stochastic systems with Markovian jump due to their wide existence in the real world, in applications of some mathematical and financial fields, for example, control in LQ models does not only affect the drift component of system dynamics but also affects the diffusion component[17]. For stochastic systems, many iterative algorithms were proposed based on the standard Lyapunov equation, which was direct copies of the deterministic systems, see [18, 19]. In this way, only the drift term information of stochastic systems is used, while the diffusion part information which means noise disturbance and refers to the random characteristic, is not fully utilized. Besides, it is not applicable when the diffusion term also contains control input by using the standard Lyapunov-type iterative methods.

Motivated by this point, we would like to develop two iterative algorithms based on the general Lyapunov equation to investigate the JLQR problem of the linear stochastic system. The main contributions are summarized as follows:

1) The model studied in this paper has multiplicative noises on both the state and control. To the best of our knowledge, there are no existing mathematical approaches to solve LQR problems of this type of system with Markovian jump. The proposed algorithms in this article provide efficient methods to design the optimal feedback control for systems of this kind.

2) Compared with some existing methods to solve CAREs problems[19], the algorithms in this article are based on the general Lyapunov-based iterative equations, which reflects a more random characteristic of the models. Besides, one of the proposed algorithms presented greatly accelerates the convergence speed compared with the other one.

3)The stochastic stability and convergence of the developed algorithms are presented and proved. Besides, the validity of the designed methods is confirmed by a relative numerical example.

Throughout this paper, unless otherwise specified, we will use the following notations. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets. The symbols A^{T} and $\|A\|$ denote the transpose and matrix norm of square matrix A, respectively. $\otimes$ is the Kronecker's tensor product and $\mathbb{E}$ stands for the mathematical expectation. If $A = (a_{ij})_{m\times n} = (\alpha_1, \ldots, \alpha_n)$, define $\mathrm{vec}(A) = (\alpha_1^{\mathrm{T}}, \alpha_2^{\mathrm{T}}, \ldots, \alpha_n^{\mathrm{T}})^{\mathrm{T}}$, which is a column vector formed by stacking the columns of A on the top of one another. The matrix relations $A > 0$ and $A \geq 0$ imply that A is positive definite and positive semi-definite, respectively.

The rest of the paper is organized as follows. In Section 2, we introduce the model descriptions concerning Markovian jump linear system. Besides, some definitions and lemmas regarding the system are given. In Section 3, the design methods and main results of the proposed iterative algorithms are given in the sequel, and the correspondent convergence properties are proved. In Section 4, a simulation example is given to illustrate the theoretical analysis. And finally, a conclusion is given in Section 5.

2. Problem formulation

2.1. Problem description

Consider the following linear stochastic system with Markovian jump

$$\begin{cases} \mathrm{d}x(t) = [A(r(t))x(t) + B(r(t))u(t)]\mathrm{d}t \\ \ + [C(r(t))x(t) + D(r(t))u(t)]\mathrm{d}w(t),\ t \geq 0, \\ x(0) = x_0, r(0) = r_0, t_0 = 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state with the initial value x_0; $u(t) \in \mathbb{R}^m$ is the control input; $A(r(t)), C(r(t)) \in \mathbb{R}^{n\times n}$ and $B(r(t)), D(r(t)) \in \mathbb{R}^{n\times m}$ are real value matrices with compatible dimension. The existence and uniqueness of the solution are guaranteed naturally . $w(t)$ is a one-dimensional standard Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, P)$; $r(t)$, taking its values in a discrete set $\mathbb{S} = \{1, 2, \cdots, N\} \in \mathbb{N}$. r_0 represents the initial mode, which is an $\mathbb{S}$-valued $\mathcal{F}_{t_0}$ measurable random variable. The solution of the equation through $(0, x_0, r_0)$ is denoted by $x(t_0, x_0, r_0)$ simply. Besides, the Markovian stochastic process $r(t)$ is assumed to be independent of the Wiener Process $w(t)$.

For the linear stochastic systems with Markovian jump (1), the transition probabilities of each mode satisfy:

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } j \neq i, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } j = i, \end{cases}$$

where $\Delta > 0$, $\lim\limits_{\Delta\to\infty} o(\Delta)/\Delta = 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to $t + \Delta$ if $j \neq i$ and $\gamma_{ii} = -\sum_{j=1, j\neq i}^{N} \gamma_{ij}$. For convenience, when $r(t) = i, A(r(t)), B(r(t)),\ C(r(t)), D(r(t)), Q(r(t)), R(r(t))$ are denoted as A_i, B_i, C_i, D_i, Q_i, R_i, respectively.

Let the initial values x_0 and r_0 be independent. For a given feedback control policy $u(t) = -K(r(t))x(t)$, we define the following cost function as:

$$\begin{aligned} & V(x(t)) \\ & = \mathbb{E}\Big\{ \int_{t_0}^{+\infty} [x^{\mathrm{T}}(t)Q(r(t))x(t) + u^{\mathrm{T}}(t)R(r(t))u(t)]\mathrm{d}t | t_0, x(t_0), r(t_0) \Big\}, \end{aligned} \tag{2}$$

The optimal control problem for the linear stochastic system with Markovian jump (1) is to find a mode-dependent admissible feedback control policy, which minimizes the performance index, i.e.

$$
\begin{aligned}
&V^*(x(t)) = \min_{u(t)} V(x(t)) \\
&= \min_{u(t)} \mathbb{E}\Big\{ \int_0^{+\infty} [x^{\mathrm{T}}(t)Q(r(t))x(t) + u^{\mathrm{T}}(t)R(r(t))u(t)]\mathrm{d}t | t_0 = 0, x_0, r_0 \Big\},
\end{aligned} \tag{3}
$$

where $Q(r(t)) \in \mathbb{R}^{n\times n} > 0, R(r(t)) \in \mathbb{R}^{m\times m} > 0$. In what follows, we denote that

$$\hat{A}_i = A_i + 0.5\gamma_{ii} I, \tag{4}$$

and

$$
\begin{cases}
M_i = {B_i}^{\mathrm{T}} P_i + {D_i}^{\mathrm{T}} P_i C_i, \\
N_i = R_i + {D_i}^{\mathrm{T}} P_i D_i, \\
K_i = -(R_i + D_i^{\mathrm{T}} P_i D_i)^{-1}({B_i}^{\mathrm{T}} P_i + D_i^{\mathrm{T}} P_i C_i),
\end{cases} \tag{5}
$$

for $i \in \mathbb{S}$, where $P_i \in \mathbb{R}^{n\times n} > 0$.

2.2. Preliminaries

Before the algorithms are formally proposed, it is necessary to ensure the stability of the system, otherwise, it doesn't make any sense under the premise of instability. Therefore, the stability definition of the system and the conditions needed, are given as follows.

Definition 1. *[20, 21] The stochastic system with Markovian jump (1) is called to be stabilizable in mean square, if there exists an admissible control $u(t) = K(r(t)x(t))$ such that the following closed-loop system*

$$
\begin{cases}
\mathrm{d}x(t) = (A(r(t)) + B(r(t))K(r(t)))x(t)\mathrm{d}t \\
\qquad\qquad + (C(r(t)) + D(r(t))K(r(t)))x(t)\mathrm{d}w(t),\ t \geq 0, \\
x(0) = x_0, r(0) = r_0,
\end{cases} \tag{6}
$$

is stable in mean square, i.e. $\lim_{t\to\infty} E[x(t)^T x(t)] = 0$, where $K(r(t)) \in \mathbb{R}^{m\times n}$ is a mode-dependent matrix.

Theorem 1. *System (1) is stabilizable in mean square if and only if ,for each mode $i \in \mathbb{S}$, there exists a control law $u(t) = L_i x(t)$ such that for any given* *positive-definite matrix J_i, the unique set of solutions H_i of the following coupled equations*

$$\begin{aligned}&(\hat{A}_i + B_i L_i)^T H_i + H_i(\hat{A}_i + B_i L_i)\\&(C_i + D_i L_i)^T H_i (C_i + D_i L_i) + \sum_{j=1, j\neq i}^{N} \gamma_{ij} H_j = -J_i,\end{aligned} \tag{7}$$

are positive definite.

Proof. The proof process of this Theorem is roughly the same as that of Theorem 1 in [22], which is omitted here.

Lemma 1. *[23] If the stochastic system with Markovian jump (1) with $Q_i > 0, R_i > 0$ for any $i \in \mathbb{S}$ is stabilizable in mean square, there exists a unique definite stabilizing solution P_i to the following CARE:*

$$\hat{A}_i^T P_i + P_i \hat{A}_i + C_i^T P_i C_i + Q_i + \sum_{j=1, j\neq i}^{N} \gamma_{ij} P_j - M_i^T N_i^{-1} M_i = 0, \tag{8}$$

for each $i \in \mathbb{S}$, where M_i, N_i are defined in (5). Then, the optimal feedback control policy can be determined by

$$u(t) = -(R_i + D_i^T P_i D_i)^{-1}(B_i^T P_i + D_i^T P_i C_i)x(t) \tag{9}$$

when $r(t) = i$, and P_i is the unique positive definite solution of (8).

On the premise that system (1) is stabilizable in mean square, then the optimization of the corresponding JLQR problem can be transformed into finding the unique solution set of a certain type of coupled algebraic Riccati equations, as described in Lemma 1.

The following lemma about the stability of matrix pairs is given to establish a relation with the existence of solutions of matrix equations and to infer the stability of corresponding systems.

Lemma 2. *[24] The following statement is equivalent:*
(1) the matrix pair (F, G) is stabilizable in mean square;
(2) for the following Lyapunov-type equation:

$$FX + F^T X + G^T X G = -Y, \tag{10}$$

if $Y > 0$(respectively, $Y \geq 0$), then $X \geq 0$(respectively, $Y \geq 0$).

Remark 1. *Different from standard Lyapunov equation, i.e.*

$$XF + F^T X = -Y,$$

which is derived from the linear deterministic system, (10) is called the general Lyapunov equation. Obviously, for the linear stochastic systems, due to the influence of the diffusion term ($Cx\mathrm{d}w(t)$), the standard Lyapunov equation is not so adaptable.

Based on the form of general Lyapunov equation (10), and with (5) , relation (8) can be rewritten as

$$\begin{aligned}&(\hat{A}_i + B_i K_i)^{\mathrm{T}} P_i + P_i(\hat{A}_i + B_i K_i) + (C_i + D_i K_i)^{\mathrm{T}} P_i (C_i + D_i K_i) \\ &= -Q_i - \sum_{j=1, j\neq i}^{N} \gamma_{ij} P_j - K_i^{\mathrm{T}} R_i K_i,\end{aligned} \tag{11}$$

as system(1) is stabilizable in mean square has been assumed, and P_i is the unique positive definite solution of (11) for $\forall i \in \mathbb{S}$, $(\hat{A}_i + B_i K_i; C_i + D_i K_i)$ is stable in mean square according to Theorem 1 and Lemma 2. Based on (11), two iterative algorithms are given in the next section.

3. Main result

In this section, two general Lyapunov-based iterative algorithms named Algorithm 1 and Algorithm 2, respectively, are proposed to design the feedback control of system(1), which is, equivalent to solving the corresponding CAREs. Algorithm 1 is a general Lyapunov-based iterative algorithm, while Algorithm 2 generalizes Algorithm 1 to the one with a tunable factor added. The convergence and monotonicity of the two algorithms have been discussed in detail.

3.1. General Lyapunov-based algorithm

In this subsection, a general Lyapunov-based algorithm named Algorithm 1 is discussed. Notice that CARE(8) is coupled and nonlinear, it is difficult to solve directly, especially for high dimensional cases. To tackle this problem, through the idea of decoupling, an iterative equation is proposed as algorithm (12). From Algorithm 1(12), the coupling terms of CARE(8) are represented by information of the previous iteration, which can be treated as constant terms.

Besides, it is observed from algorithm (12) that the estimate $P_i(m+1)$ for the solution P_i is updated by using the estimate at the m-step. By performing such iteration, sequences of related solutions $\{P_i(m)\}$ can be obtained, and the limit of $P_i(m)$ is P_i as $m \to +\infty$, then the solutions to CARE(8) are numerically approximated. The convergence properties of Algorithm 1 will be discussed later.

In the proposed algorithm, to ensure its convergence, the initial value $P_i(0)$ should satisfy

$$\begin{aligned}
&(\hat{A}_i + B_iK_i(0))^{\mathrm{T}}P_i(0) + P_i(0)(\hat{A}_i + B_iK_i(0)) \\
&+ (C_i + D_iK_i(0))^{\mathrm{T}}P_i(0)(C_i + D_iK_i(0)) + Q_i \\
&+ \sum_{j=1,j\neq i}^{N} \gamma_{ij}P_j(0) + K_i(0)^{\mathrm{T}}R_iK_i(0) + \varepsilon = 0,
\end{aligned} \tag{15}$$

where ε is a arbitrary positive constant, $K_i(0) = -(R_i+D_i^{\mathrm{T}}P_i(0)D_i)^{-1}(B_i^{\mathrm{T}}P_i(0)+D_i^{\mathrm{T}}P_i(0)C_i), P_i(0) > 0$, then $K_i(0)$ stabilizes system (1) according to Lemma 1. According to Theorem 1 and Lemma 2, we can be conclude that $(\hat{A}_i + B_iK_i(0), C_i + D_iK_i(0))$ is stable. Let (12) with $m = 0$, that is

$$\begin{aligned}
&(\hat{A}_i + B_iK_i(0))^{\mathrm{T}}P_i(1) + P_i(1)(\hat{A}_i + B_iK_i(0)) \\
&+ (C_i + D_iK_i(0))^{\mathrm{T}}P_i(1)(C_i + D_iK_i(0)) \\
&= -Q_i - K_i^{\mathrm{T}}(0)R_iK_i(0) - \sum_{j=1,j\neq i}^{N} \gamma_{ij}P_j(0),
\end{aligned} \tag{16}$$

and subtracting (15) from (16), we can easily have

$$\begin{aligned}
&(\hat{A}_i + B_iK_i(0))^{\mathrm{T}}(P_i(1) - P_i(0)) + (P_i(1) - P_i(0))(\hat{A}_i + B_iK_i(0)) \\
&+ (C_i + D_iK_i(0))^{\mathrm{T}}(P_i(1) - P_i(0))(C_i + D_iK_i(0)) \\
&= \varepsilon,
\end{aligned} \tag{17}$$

as $\varepsilon > 0$, according to Lemma 2 $P_i(1) - P_i(0) < 0$, i.e. $P_i(0) > P_i(1)$ can be

Algorithm 1

Step 1: Given the proper initial values $P_i(0)$, $i \in \mathbb{S}$ such that $K_i(0)$ stabilizes system (1), and let $m = 0$;

Step 2: Solving the following N decoupled general Lyapunov equations for $P_i(m+1), i \in \mathbb{S}$:

$$
\begin{aligned}
&(\hat{A}_i + B_i K_i(m))^{\mathrm{T}} P_i(m+1) + P_i(m+1)(\hat{A}_i + B_i K_i(m)) \\
&+ (C_i + D_i K_i(m))^{\mathrm{T}} P_i(m+1)(C_i + D_i K_i(m)) \\
&= -Q_i - K_i^{\mathrm{T}}(m) R_i K_i(m) - \sum_{j=1, j\neq i}^{N} \gamma_{ij} P_j(m),
\end{aligned}
\tag{12}
$$

where

$$
K_i(m) = -(R_i + D_i^{\mathrm{T}} P_i(m) D_i)^{-1} \times (B_i^{\mathrm{T}} P_i(m) + D_i^{\mathrm{T}} P_i(m) C_i);
$$

Step 3: Let $m = m + 1$ and compute

$$
\delta(m) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \|f(P_i)\|},
\tag{13}
$$

where

$$
\begin{aligned}
&f(P_i(m)) = \hat{A}_i^{\mathrm{T}} P_i(m) + P_i(m)\hat{A}_i + C_i^{\mathrm{T}} P_i(m) C_i + Q_i \\
&+ \sum_{j=1, j\neq i}^{N} \gamma_{ij} P_j(m) - M_i^{\mathrm{T}}(m) N_i^{-1}(m) M_i(m);
\end{aligned}
\tag{14}
$$

and

$$
\begin{cases}
M_i(m) = {B_i}^{\mathrm{T}} P_i(m) + {D_i}^{\mathrm{T}} P_i(m) C_i, \\
N_i(m) = R_i + {D_i}^{\mathrm{T}} P_i(m) D_i,
\end{cases}
$$

Step 4: If $\delta(m) \leq \varepsilon$ (ε is a given small nonnegative constant), then stop and output $\{P_i(m+1), i \in \mathbb{S}\}$ as the solution $\{P_i, i \in \mathbb{S}\}$ of (8). Else, go to Step 2.

concluded. Subtracting (8) from (15), we have

$$
\begin{aligned}
&(\hat{A}_i + B_iK_i(0))^{\mathrm{T}}(P_i(0) - P_i) + (P_i(0) - P_i)(\hat{A}_i + B_iK_i(0)) \\
&+ (C_i + D_iK_i(0))^{\mathrm{T}}(P_i(0) - P_i)(C_i + D_iK_i(0)) + \sum_{j=0, j\neq i}^{N} \gamma_{ij}(P_j(0) - P_j) \\
&= -\varepsilon - (K_i - K_i(0))^{\mathrm{T}}N_i(K_i - K_i(0)),
\end{aligned} \tag{18}
$$

According to Theorem 1, the solution of (18) is positive definition, i.e. $P_i(0) - P_i > 0$.

3.2. Improved general Lyapunov-based algorithm

As mentioned above, when estimating the value of $P_i(m+1)$, only the information in the m-th step, i.e. $P_i(m)$, is used to represent the coupling term. In fact, in the (m+1)-th step of the iteration, estimations $P_j(m+1), j \in \{1, 2, \ldots, i-1\}$ have already been obtained before estimation $P_i(m+1)$ is calculated. Therefore, this part of information is available to update the estimation $P_i(m+1)$ in the $(m+1)$-th step. To take full advantage of the previous estimation results, an improved general Lyapunov-based iterative algorithm named Algorithm 2 (19) is proposed. Besides, a tunable factor is added based on Algorithm 1, which is described in detail in chart Algorithm 2.

In Algorithm 1, under proper initial value $P_i(0)$, $K_i(0)$ stabilizes system (1), and then it follows that $P_i(0) > P_i(1)$ and $P_i(0) > P_i$, which are necessary conditions to determine the monotone convergence of solution sequences regarding to iterative equation (12). This conclusion also applies to Algorithm 2.

For Algorithm 2, iterative step (19) for some $i \in \mathbb{S}$, the estimation value $P_i(m+1)$ is updated by the weighted average of the estimated values of the last step and current step. Additionally, it is easily noted that (19) can degenerate to (12) with $\omega = 1$, which means that Algorithm 2 is a generalization of Algorithm 1.

3.3. The convergence properties

In this subsection, the convergence property of algorithm (19) will be analyzed. For this reason, the boundedness and monotonicity for the solution sequences generated by (19) will be established in sequence. Since Algorithm 2 is a generalization of Algorithm 1, here only the convergence of Algorithm

Algorithm 2

Step 1: Given the proper initial values $P_i(0)$, $i \in \mathbb{S}$ such that $K_i(0)$ stabilizes system (1), and let $m = 0$;

Step 2: Solving the following N decoupled general Lyapunov equations for $P_i(m+1), i \in \mathbb{S}$:

$$\begin{aligned}
&(\hat{A}_i + B_i K_i(m))^{\mathrm{T}} P_i(m+1) + P_i(m+1)(\hat{A}_i + B_i K_i(m)) \\
&+ (C_i + D_i K_i(m))^{\mathrm{T}} P_i(m)(C_i + D_i K_i(m)) \\
&= -Q_i - K_i^{\mathrm{T}}(m) R_i K_i(m) - \sum_{j=i+1}^{N} \gamma_{ij} P_j(m) \\
&- \sum_{j=0}^{i-1} \gamma_{ij}[(1-\omega)P_j(m+1) + \omega P_j(m)],
\end{aligned} \tag{19}$$

where

$$K_i(m) = -(R_i + D_i^{\mathrm{T}} P_i(m) D_i)^{-1} \times (B_i^{\mathrm{T}} P_i(m) + D_i^{\mathrm{T}} P_i(m) C_i);$$

Step 3: Let $m = m + 1$ and compute

$$\delta(m) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \|f(P_i)\|}, \tag{20}$$

where

$$\begin{aligned}
f(P_i(m)) &= \hat{A}_i^{\mathrm{T}} P_i(m) + P_i(m)\hat{A}_i + C_i^{\mathrm{T}} P_i(m) C_i + Q_i \\
&+ \sum_{j=1, j\neq i}^{N} \gamma_{ij} P_j(m) - M_i^{\mathrm{T}}(m) N_i^{-1}(m) M_i(m);
\end{aligned} \tag{21}$$

and

$$\begin{cases} M_i(m) = {B_i}^{\mathrm{T}} P_i(m) + {D_i}^{\mathrm{T}} P_i(m) C_i, \\ N_i(m) = R_i + {D_i}^{\mathrm{T}} P_i(m) D_i, \end{cases}$$

Step 4: If $\delta(m) \leq \varepsilon$ (ε is a given small nonnegative constant), then stop and output $\{P_i(m+1), i \in \mathbb{S}\}$ as the solution $\{P_i, i \in \mathbb{S}\}$ of (8). Else, go to Step 2.

2 will be proved. First, the boundedness property of the algorithm (19) is obtained by the following Lemma.

Lemma 3. *If system (1) is stabilizable in mean square and* $K_i(0)$ *stabilizes the system, then for any integer* $m \geq 0, i \in \mathbb{S}$*, the sequences* $\{P_i(m)\}$ *generated by (19) with* $0 \leq \omega \leq 1$ *have the following properties:*

(1) $P_i(m) > P_i$*;*

(2) $(\hat{A}_i + B_i K_i(m), C_i + D_i K_i(m))$ *is stable.*

Proof. We prove the lemma by mathematical induction. By subtracting (8) from (19), we obtain the following relation

$$\begin{aligned}&(\hat{A}_i + B_i K_i(m))^{\mathrm{T}} \delta P_{i,m+1} + \delta P_{i,m+1} (\hat{A}_i + B_i K_i(m)) \\ &+ (C_i + D_i K_i(m))^{\mathrm{T}} \delta P_{i,m+1} (C_i + D_i K_i(m)) \\ &= I_i(m+1),\end{aligned} \tag{22}$$

with

$$\delta P_{i,m+1} = P_i(m+1) - P_i, \forall i \in \mathbb{S}, m \geq 0,$$

and

$$\begin{aligned}I_i(m+1) = &-(K_i(m) - K_i)^{\mathrm{T}} N_i (K_i(m) - K_i) \\ &- \sum_{j=1}^{i-1} \gamma_{ij} [(1-\omega)\delta P_{j,m+1} + \omega \delta P_{j,m}] - \sum_{j=i+1}^{N} \gamma_{ij} \delta P_{j,m}.\end{aligned}$$

Rewriting (22), we have the following relation

$$\begin{aligned}&(\hat{A}_i + B_i K_i(m+1))^{\mathrm{T}} \delta P_{i,m+1} + \delta P_{i,m+1} (\hat{A}_i + B_i K_i(m+1)) \\ &+ (C_i + D_i K_i(m+1))^{\mathrm{T}} \delta P_{i,m+1} (C_i + D_i K_i(m+1)) \\ &= W_i(m+1),\end{aligned} \tag{23}$$

with

$$\begin{aligned}&W_i(m+1) \\ &= -(K_i(m+1) - K_i(m))^{\mathrm{T}} N_i(m+1) (K_i(m+1) - K_i(m)) \\ &\quad - (K_i(m+1) - K_i)^{\mathrm{T}} N_i (K_i(m+1) - K_i) \\ &\quad - \sum_{j=1}^{i-1} \gamma_{ij} [(1-\omega)\delta P_{j,m+1} + \omega \delta P_{j,m}] - \sum_{j=i+1}^{N} \gamma_{ij} \delta P_{j,m}.\end{aligned}$$

Firstly, since the case with $m = 0$ is preconditioned, conditions (1), (2) of this lemma at the starting position, namely, $m = 1$, need to be proved. For the case with $m = 1$, in view of $N_i, N_i(1) > 0$ and $0 \leq \omega \leq 1$, it can be obtained that $I_i(1), W_i(1) < 0$. For (22), based on $I_i(1) < 0$ and $(\hat{A}_i + B_iK_i(0), C_i + D_iK_i(0))$ is stable, $\delta P_{i,1} > 0$ can be derived according to Lemma 2, that is, $P_i(1) - P_i > 0$. As the positive solution of (23) $\delta P_{i,1}$ already exists as mentioned above, $(\hat{A}_i + B_iK_i(1), C_i + D_iK_i(1))$ is stable can be concluded from (23) according to Lemma 2. Then the conclusions (1)-(2) of this lemma hold for $m = 1$.

Next, suppose the conclusions (1)-(2) of this lemma hold for $m = k$, that is $P_i(k) > P_i$ and $(\hat{A}_i + B_iK_i(k), C_i + D_iK_i(k))$ is stable. For (22) and (23) with $m = k$, in view of $P_i(k) > P_i$, we have $I_i(k+1), W_i(k+1) < 0$. For (22), $\delta P_{i,k+1} > 0$ can be easily derived in view of the already obtained conclusions, i.e. $I_i(k+1) < 0$ and $(\hat{A}_i + B_iK_i(k), C_i + D_iK_i(k))$ is stable. And then, we can prove that $(\hat{A}_i + B_iK_i(k+1), C_i + D_iK_i(k+1))$ is stable according to (23) based on Lemma 2. Namely the conclusion also holds for $m = k+1$. By the mathematical induction, our conclusion holds for arbitrary integer $m \geq 0$. The proof is complete.

Next, a lemma is given to investigate the monotonicity property of Algorithm 2.

Lemma 4. *If system (1) is stabilizable in mean square and $K_i(0)$ stabilizes system (1),then the m-th step solution $P_i(m)$ generated by (19) with $0 \leq \omega \leq 1$ has the following property:*

$$P_i(m) > P_i(m+1), \tag{24}$$

for any integer $m \geq 0$.

Proof. Relation (19) can be rewritten as

$$\begin{aligned}&(\hat{A}_i + B_iK_i(m))^{\mathrm{T}}\delta P_i(m+1) + \delta P_i(m+1)(\hat{A}_i + B_iK_i(m))\\&+ (C_i + D_iK_i(m))^{\mathrm{T}}\delta P_i(m+1)(C_i + D_iK_i(m))\\&= T_i(m+1).\end{aligned} \tag{25}$$

with

$$\begin{aligned}&T_i(m+1)\\&= (K_i(m) - K_i(m-1))^{\mathrm{T}}N_i(m)(K_i(m) - K_i(m-1))\\&- \sum_{j=1}^{i-1}\gamma_{ij}[(1-\omega)\delta P_j(m+1) + \omega\delta P_j(m)] - \sum_{j=i+1}^{N}\gamma_{ij}\delta P_j(m).\end{aligned}$$

and

$$\delta P_i(m+1) = P_i(m+1) - P_i(m), \forall i \in \mathbb{S}, m \geq 0,$$

As $(\hat{A}_i + B_i K_i(m), C_i + D_i K_i(m))$ is stable has been prove in Lemma 3, by using (25) the conclusion of this lemma can be proved with the similar proof procedure of Lemma 3, we omit it here.

Based on the conclusions in Lemma 3 and 4 and the convergence result in [25], the following theorem about the convergence property of the proposed Algorithm 2 is given.

Theorem 2. *Assume that Assumption 1 is satisfied and system (1) is stabilizable in meas square. Let* $\{P_i,\ i \in \mathbb{S}\}$ *be the unique positive definite solution of the CARE (8), then the solution* $P_i(m)$ *generated by Algorithm 2 with* $\omega \in [0, 1]$ *converges to the unique positive definite solution of the CARE (8), that is,* $\lim\limits_{m\to\infty} P_i(m) = P_i, \forall i \in \mathbb{S}$.

Proof. Based on the the conclusions of Lemma 3 and 4, we have $P_i(0) > P_i(1) \cdots P_i(m) > P_i(m+1) > \cdots > P_i$ for each $i \in \mathbb{S}$. According to the limit theory of the sequence, $\lim\limits_{m\to\infty} P_i(m) = P_i, \forall i \in \mathbb{S}$. The proof is complete.

Remark 2. *In Theorem 1, it is only proved theoretically that (19) is convergent under the condition* $0 \leq \omega \leq 1$. *However, for practical applications, two questions remain to be further investigated on the selection of parameter* ω*: 1. an exact range of parameter* ω *which is the necessary and sufficient condition for the convergence of Algorithm 2 is to be given; 2. an approach for choosing a parameter* ω *such that Algorithm 2 converges the fastest is to be established. We would expect to investigate these two questions in the future.*

Remark 3. *Algorithm 1(12) and Algorithm 2(19) can proceed on condition that the following conditions are satisfied:*
(1) $R_i + {D_i}^{\mathrm{T}} P_i(m) D_i, \forall i \in \mathbb{S}, \forall m \in \{0, 1, 2, \cdots\}$ *is reversible, or* $K_i(m), i \in \mathbb{S}$ *doesn't make any sense;*
(2) The unique solutions of Algorithm 1(12) and Algorithm 2(19) exist for any iteration.

In the conclusion of Lemma 3 (1), we have already obtained that $P_i(m) > P_i > 0, i \in \mathbb{S}$*, thus for any integer* $m \geq 0, R_i + {D_i}^T P_i(m) D_i$ *is invertible.*

Besides, the condition (2) of this remark is satisfied naturally according to that $(\hat{A}_i + B_iK_i(m), C_i + D_iK_i(m))$ is stable for $i \in \mathbb{S}, \forall m \geq 0$ according to Lemma 2.

4. Illustrative Example

In this section, a numerical example is used to show the efficiency of the proposed algorithms. In order to characterize the convergence performance of the proposed algorithm clearly, the iterative error versus the iteration step m is defined as $log_{10}\delta(m)$, where $\delta(m)$ is defined in (20).

Example 1. Let $n = 3, x = [x_1, x_2, x_3]^T \in \mathbb{R}^2$. Consider the CARE (8) with transition rate matrix

$$\Gamma = \begin{bmatrix} -0.5 & 0.5 \\ 2.5 & -2.5 \end{bmatrix}, \tag{26}$$

and the following system parameter matrices

$$A_1 = \begin{bmatrix} -2.9134 & -1.6785 & -0.9649 \\ -0.5010 & -2.5469 & -1.0006 \\ -0.0975 & -0.9575 & -1.9706 \end{bmatrix}, A_2 = \begin{bmatrix} -1.9572 & 0.1419 & 0.7922 \\ 0.4854 & -1.4218 & 1.9595 \\ 0.8003 & -0.9157 & -2.6557 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.8357 \\ -0.8491 \\ -0.9340 \end{bmatrix}, B_2 = \begin{bmatrix} -0.6787 \\ -0.7577 \\ -0.7431 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.3922 & 0 & 0 \\ 0 & -0.6555 & 0 \\ 0 & 0 & -0.4712 \end{bmatrix}, C_2 = \begin{bmatrix} 0.7060 & 0 & 0 \\ 0 & -0.0318 & 0 \\ 0 & 0 & 0.2769 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.4462 \\ -0.3971 \\ -0.5235 \end{bmatrix}, D_2 = \begin{bmatrix} -0.6948 \\ -0.3171 \\ -0.9502 \end{bmatrix},$$

The positive definite matrices Q_i and R_i related to the cost function (2), are all chosen as identity matrices with appropriate dimension. Besides, matric pairs$(A_i, B_i), (C_i, D_i)$ are controllable for $\forall i \in \mathbb{S}$.

First, Algorithm 2 is used to approach the unique positive definite solution of the CARE (8). What calls for special attention is that the initial value $P_i(0)$ should chosen to satisfy (15), so here $P_i(0)$ can be chosen as identity matrices with appropriate dimension then the following inequality holds:

$$
\begin{aligned}
&(\hat{A}_i + B_iK_i(0))^{\mathrm{T}}P_i(0) + P_i(0)(\hat{A}_i + B_iK_i(0)) \\
&+ (C_i + D_iK_i(0))^{\mathrm{T}}P_i(0)(C_i + D_iK_i(0)) + Q_i \\
&+ \sum_{j=1, j\neq i}^{N} \gamma_{ij}P_j(0) + K_i(0)^{\mathrm{T}}R_iK_i(0) < 0.
\end{aligned}
$$

When the initial value set $\{P_i(0)\}$ is chosen according to (15), with a range of specially chosen parameters, the associated convergence curves can be plotted as the convergence curves are depicted as Fig.1, which Fig.1demonstrates the convergence of Algorithm 2 in a specific range $\omega \in [0, 1]$. Though not directly proved in this paper, the real convergence region should be larger the what has been proved in this paper. When $\omega = 0$, the convergent performance of Algorithm 2 is the best.

Fig.2 shows the comparison of Algorithm 1, Algorithm 2, and algorithm presented in [12]. From this figure, both Algorithm 1 and Algorithm 2 have a better convergence effect than the one in [12]. Besides, the solution sequence $\{P_i(m)\}$ generated by Algorithm 2 is monotonic, which can be viewed by observing the eigenvalues of the differences $\{P_i(m+1)-P_i(m)\}$, where integer $m \geq 0$. For special purposes, ω is set as 1 and the result can be seen from Table 1.

By using Algorithm 2, the unique positive definite solution of the CARE (8) is given as

$$
P_1 = \begin{bmatrix} 0.2208 & -0.1272 & -0.0275 \\ -0.1272 & 0.4179 & -0.1747 \\ -0.0275 & -0.1747 & 0.3816 \end{bmatrix}, P_2 = \begin{bmatrix} 0.3084 & -0.1544 & -0.0556 \\ -0.1544 & 0.5010 & -0.0759 \\ -0.0556 & -0.0759 & 0.2435 \end{bmatrix}.
$$

5. Conclusion

In this paper, two iterative algorithms named Algorithm 1 and Algorithm 2 respectively based on general Lyapunov equations, are proposed to obtain the unique positive definite solution of the CAREs for linear stochastic systems with Markovian jump. Algorithm 2 generalizes Algorithm 1 to

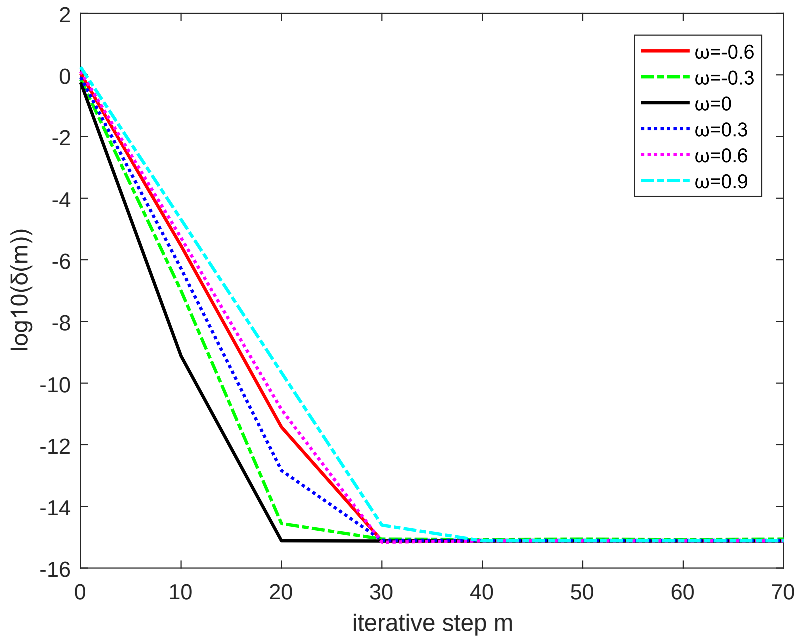

Fig. 1: Comparison convergence rate for the proposed Algorithm 2 with different parameters.

Table 1: Monotonicity of the solutions of Algorithm 2 with $\omega = 1$

m	$\lambda(P_1(m+1) - P_1(m))$	$\lambda(P_2(m+1) - P_2(m))$
0	$(-0.8209, -0.6001, -0.2837)$	$(-0.0904, -0.4207, -0.6313)$
1	$(-0.0195, -0.0539, -0.0906)$	$(-0.3216, -0.2097, -0.1418)$
2	$(-0.0111, -0.0459, -0.0316)$	$(-0.0394, -0.0217, -0.0140)$
3	$(-0.0093, -0.0023, -0.0009)$	$(-0.0283, -0.0034, -0.0130)$
4	$(-0.0049, -0.0002, -0.0023)$	$(-0.0051, -0.0013, -0.0003)$
5	$(-0.0011, -0.0000, -0.0002)$	$(-0.0030, -0.0001, -0.0010)$
$\cdots$	$\cdots$	$\cdots$
20	$10^{-9}(-0.2064, -0.0053, -0.0938)$	$10^{-8}(-0.0049, -0.0015, 0.0115)$

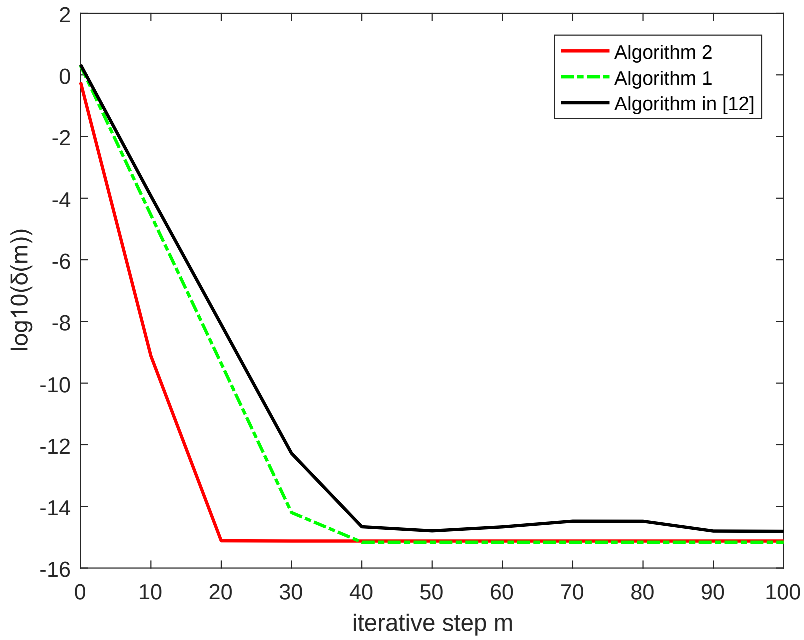

Fig. 2: Comparison convergence rate for the proposed algorithms and standard Lyapunov-based algorithm.

the one with adding a tunable factor, and it has been verified that with the appropriate initial value, the solution sequence obtained by Algorithm 2 (or Algorithm 1) monotonically converges to the unique positive definite solution. Besides, the convergence speed of Algorithm 2 are greatly improved compared to the standard Lyapunov-based iterative algorithm.

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