Solution of two-phase cylindrical inverse Stefan problem by using special functions

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Abstract

In this work two-phase Stefan problem for the cylindrical heat equation is considered. One of the phase turns to zero at initial time. In this case, it is difficult to solve by radial heat polynomials because the equations are singular. The solution is represented in linear combination series of special functions Laguerre polynomial and confluent hyper-geometric function. The free boundary is given and heat flux is found. The numerical and approximate test problem is compared graphically. The undetermined coefficients are founded. The convergence of series proved.

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Introduction

The cylindrical heat equation is an important in mathematical modeling of heat transfer in bodies which cylindrical domain. The solution, with another words temperature distribution, in such model can take form of series for special function (Laguerre polynomials). In the developement heat processes models the partial differential equations play necessary role (2014; Zvyagin V, 2015; Slota D. Homotopy perturbation method for solving the two-phase inverse Stefan problem. Numer. Heat Transfer. Part A: Appl, 2011). The free boundaries theory takes the big process in the last half century. We refer to Chen et al.(Chen G-Q & future developments. Phil. Trans R. Soc. 2015; A373:20140285-1-20140285-8, 2015) and Friedman (boundary problems in biology. Phil. Trans. R. Soc. 2015; A373:20140368-1-20140368-16, 2015; boundary problems for parabolic equations I. Melting of solids. J. Math. Mec. 1959; 8:499-517, 1959) literatures to realize some models which can be expressed as free boundary problems. In process of heat arcing the phase transformation takes place, therefore we consider Stefan-type problems (Bermidez A, 2006; Rubinstein L. I. The Stefan problem. Trans. Math. Monogr. Vol. 27, 1971). Present study is devoted to such Stefan problem in which temperature functions considered in form of special functions with unknown coefficients which have close link introduced by P.C. Rosenbloom and D.V. Widder (Rosenbloom et al., 1959). To represent solution form of the problem we consider the following equation

$$x\frac{d^2\phi}{dx^2}+\big(\frac{\nu+1}{2}-x\big)\frac{d\phi}{dx}+\frac{\beta}{2}\phi=0,\quad \nu=0,\quad -\infty<\beta<\infty.$$

(1)

It is well known that this equation has two linearly independent solutions

$$\phi_1(x) = \Phi\big(-\frac{\beta}{2}, \frac{\nu+1}{2}; x\big), \quad \phi_2(x) = x^{\frac{1-\nu}{2}}\Phi\big(\frac{1-\beta-\nu}{2}, \frac{3-\nu}{2}; x\big),$$

(2)

where $\Phi(a, b; x)$ is the confluent (degenerate) hypergeometric function. Setting $T(z) = \phi(x)$, where $x = -z^2$, one can find that $T(z)$ satisfies the equation

$$\frac{d^2T}{dz^2} + \big(\frac{\nu}{z} + 2z\big)\frac{dT}{dz} - 2\beta T(z) = 0$$

Using this equation one can check up that the function

$$\theta(z,t) = (2a\sqrt{t})^{\beta} T\left(\frac{z}{2a\sqrt{t}}\right)$$

satisfies the equation

$$\frac{\partial\theta}{\partial t} = a^2\left(\frac{\partial^2\theta}{\partial z^2} + \frac{\nu}{z}\frac{\partial\theta}{\partial z}\right).$$

(3)

Hence the functions

(4)

satisfy the equation (3). If β is an even integer, $\beta = 2n$, the function $S_{\beta,\nu}(z,t)$ can be expressed in terms of the generalized Laguerre polynomials

$$S^{(1)}_{2n,\nu}(z,t) = (4a^2t)^n\Phi\left(-n, \mu; -\frac{z^2}{4a^2t}\right) = \frac{n!\Gamma(\mu)}{\Gamma(\mu+n)}(4a^2t)^n L_n^{\mu-1}\left(-\frac{z^2}{4a^2t}\right),$$

(4)

$$S^{(2)}_{2n,\nu}(z,t) = (4a^2t)^n\left(\frac{z^2}{4a^2t}\right)^{1-\mu}\Phi\left(1-\mu-n, 2-\mu; -\frac{z^2}{4a^2t}\right) = \frac{n!\Gamma(\mu)}{\Gamma(\mu+n)}(4a^2t)^n\left(\frac{z^2}{4a^2t}\right)^{1-\mu}L_n^{\mu-1}\left(-\frac{z^2}{4a^2t}\right),$$

(4)

where $\mu = \dfrac{\nu+1}{2}$. It should be noted that this formula is valid for $\mu > 0$ only.
Using the integral representation for the degenerate hypergeometric function

$$\Phi\big(-\frac{\beta}{2}, \mu, -z^2\big) = \frac{2\Gamma(\mu)}{\Gamma(\mu+\beta/2)}\exp(-z^2)z^{-\mu+1}\int\limits_0^\infty \exp(-x^2)x^{\mu+\beta}I_{\mu+1}(2zx)dx$$

(4)

and the asymptotic formula

$$\lim_{z\to\infty}\frac{e^{-z}I_\nu(z)}{\sqrt{2\pi z}} = 1$$

it is possible to show that

$$\lim_{z\to\infty}\frac{1}{z^\beta}\Phi\big(-\frac{\beta}{2}, \mu; -z^2\big) = \frac{\Gamma(\mu)}{\Gamma\big(\mu+\frac{\beta}{2}\big)}.$$

(4)

In particular,

$$\lim_{z\to\infty}\frac{1}{z^\beta}\Phi\big(-\frac{\beta}{2}, 1; -z^2\big) = \frac{1}{\Gamma\big(1+\frac{\beta}{2}\big)}.$$

(4)

For $\nu = 1$ both functions (1) coincide:

$$S^{(1)}_{\beta,1}(z,t) = S^{(2)}_{\beta,1}(z,t) = (2a\sqrt{t})^{\beta}\Phi\biggl(-\frac{\beta}{2}, 1; -\frac{z^2}{4a^2t}\biggr).$$

In this case, the second linearly independent solution of the equation (3) is (G. Szego. Orthogonal polynomials. American Mathematical Society, 1939)

$$\phi_2(x) = \Phi\bigl(-\frac{\beta}{2}, 1; x\bigr)\ln x + \sum_{k=1}^{\infty} M_k x^k,$$

(4)

where

$$M_k =$$

$$\begin{pmatrix} k \\ -\frac{\beta}{2} \end{pmatrix} \ \frac{1}{k!}\sum_{m=0}^{k-1}\biggl(\frac{1}{m-\beta/2} + \frac{2}{m+1}\biggr).$$

Problem definition

We consider the following problem of heat transfer in solid phase $0 < r < \alpha(t)$ and in liquid phase $\alpha(t) < r < \infty$ which can be modelled with cylindrical heat equation

$$\frac{\partial\theta_1}{\partial t} = a_1^2\biggl(\frac{\partial^2\theta_1}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_1}{\partial r}\biggr), \ \ 0 < r < \alpha(t), \ \ t > 0,$$

(4)

$$\frac{\partial\theta_2}{\partial t} = a_2^2\biggl(\frac{\partial^2\theta_2}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_2}{\partial r}\biggr), \ \ \alpha(t) < r < \infty, \ \ t > 0$$

(4)

with initial conditions

$$\theta_1(0,0) = 0, \quad \alpha(0) = 0,$$

(4)

$$\theta_2(r,0) = f(r)$$

(4)

and boundary conditions

$$-\lambda_1 \frac{\partial \theta_1(\alpha(t),t)}{\partial r} = P(t) - L\gamma \frac{d\alpha}{dt},$$

(4)

$$\theta_1(\alpha(t),t) = \theta_2(\alpha(t),t) = \theta_m.$$

(4)

Stefan's condition at free boundary is

$$-\lambda_1 \frac{\partial \theta_1(\alpha(t),t)}{\partial r} = -\lambda_2 \frac{\partial \theta_2(\alpha(t),t)}{\partial r} + L\gamma \frac{d\alpha}{dt}$$

(4)

and condition at infinity is

$$\theta_2(\infty,t) = 0.$$

(4)

Problem solution

For $\beta = 2n$ we can represent solution of (4)-(4) as form of linear combinations of special functions

$$\theta_1(r,t) = \sum_{n=0}^{\infty} A_n (4a_1^2 t)^n L_n\left(-\frac{r^2}{4a_1^2 t}\right) + \sum_{n=0}^{\infty} B_n (4a_1^2 t)^n \left[\Phi\left(-n,1;-\frac{r^2}{4a_1^2 t}\right) \ln\left(\frac{r^2}{4a_1^2 t}\right) + \sum_{k=1}^{\infty} M_k \left(-\frac{r^2}{4a_1^2 t}\right)^k\right],$$

(4)

$$\theta_2(r,t) = \sum_{n=0}^{\infty} C_n (4a_2 t)^n L_n\Big(-\frac{r^2}{4a_2^2 t}\Big),$$

(4)

where $M_k =$
$\binom{k}{n} \; \frac{1}{k!} \sum_{m=0}^{k-1} \left(\frac{1}{m+n} + \frac{2}{m+1} \right)$ and $\theta_1(r,t),\ \theta_2(r,t)$ are temperatures in solid and liquid zones, θ_m is melting temperature and A_n, B_n, C_n are unknown which have to be determined. The equations (4) and (4) satisfy the problem (4)-(4), consider that function $f(r) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{(n)!} r^n$ and free boundary $\alpha(t) = \sum_{n=1}^{\infty} \alpha_n t^{n/2}$ are given. From (4) we have

$$\sum_{n=0}^{\infty} C_n \frac{r^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} r^n$$

(4)

and comparing degree of r we obtain definition for coefficient C_n as follows

$$C_n = \frac{f^{(2n)}(0)}{(2n)!} n!, \ \ n = 0, 1, 2, ...$$

(4)

The algorithm to find coefficients A_n and B_n is that from condition (4) we express coefficients A_n and by making substitution to (4) we can get coefficients B_n. At first, we take m-th derivative from (4) and (4) when $\tau = 0$, where $\tau = \sqrt{t}$.

$$\frac{\partial^m \theta_1(\alpha(\tau),\tau)}{\partial \tau^m}\bigg|_{\tau=0} = \frac{\partial^m \theta_2(\alpha(\tau),\tau)}{\partial \tau^m}\bigg|_{\tau=0} = \frac{\partial^m \theta_m}{\partial \tau^m},$$

(4)

$$-\lambda_1 \frac{\partial^m}{\partial \tau^m}\left[\frac{\partial \theta_1(\alpha(\tau),\tau)}{\partial r}\right]\bigg|_{\tau=0} = -\lambda_2 \frac{\partial^m}{\partial \tau^m}\left[\frac{\partial \theta_2(\alpha(\tau),\tau)}{\partial r}\right]\bigg|_{\tau=0} + L\gamma \frac{d^m \alpha}{d\tau^m}.$$

(4)

We use the Leibniz rule for (4) and (4), then we have

$$\left.\frac{\partial^m\left[(4a_1^2)^n\tau^2 L_n(-\delta(\tau))\right]}{\partial\tau^m}\right|_{\tau=0} = (4a_1^2)^n\frac{m!}{(m-2n)!}\left.\left[L_n(-\delta(\tau))\right]^{(m-2n)}\right|_{\tau=0}$$

and

$$\binom{m-2n}{l}\left[\Phi[-n,1;-\delta(\tau)]\right]^{(l)}\left[\ln(\delta(\tau))\right]^{(m-2n-l)}+\left.\sum_{k=1}^{\infty}M_k\left[(-\delta(\tau))^k\right]^{(m-2n)}\right|_{\tau=0},$$

where $\delta(\tau) = \frac{1}{4a_1^2}(\alpha_1+\alpha_2\tau+\alpha_3\tau^2+...)^2 = \frac{1}{4a_1^2}\left(\sum_{n=1}^{\infty}\alpha_n\tau^{n-1}\right)^2$, $i = 1,2$. Then we use Faa-di Bruno formula for taking derivative from composite function and we get the following results

$$\left.\frac{\partial^{m-2n}\left[L_n(-\delta(\tau))\right]}{\partial\tau^{m-2n}}\right|_{\tau=0} = \sum_{l=0}^{m-2n}\left[L_n(-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!},$$

(4)

$$\left.\frac{\partial^{m-2n}\left[\Phi(-n,1;-\delta(\tau))\right]}{\partial\tau^{m-2n}}\right|_{\tau=0} = \sum_{l=0}^{m-2n}\left[\Phi(-n,1;-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!},$$

(4)

$$\left.\frac{\partial^{m-2n-l}\left[\ln(\delta(\tau))\right]}{\partial\tau^{m-2n-l}}\right|_{\tau=0} = \sum_{p=0}^{m-2n-l}\left[\ln(\delta_1)\right]^{(p)}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2!b_3!...b_{m-2n-l-p+2}!},$$

(4)

$$\left.\frac{\partial^{m-2n}\left[(-\delta(\tau))^{k}\right]}{\partial\tau^{m-2n}}\right|_{\tau=0}=\sum_{l=0}^{m-2n}(-\delta_1)^{k-l}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!},$$

(4)

where $\delta_1=\frac{\alpha_1^2}{4a_i^2}$, $\delta_2=\frac{\alpha_2^2}{4a_i^2},...,\delta_{m-2n-l-p+2}=\frac{\alpha_{m-2n-l-p+2}^2}{4a_i^2}$, $i=1,2$ and $b_1,b_2,b_3,...$ satisfy the following equations

$$b_2+b_3+...+b_{m-2n-l-p+2}=m,$$

$$2b_2+3b_3+...+(m-2n-l-p+2)b_{m-2n-l-p+2}=m-2n.$$

In particular, when $m=0$ and $\tau=0$ we have the following initial coefficients

(4)

By using formulas (4)-(4) we have the next recurrent expressions for conditions (4) and (4)

$$\binom{m-2n}{l}\left[\Phi(-n,1;-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!}$$
$$\cdot\sum_{p=0}^{m-2n-l}\left[\ln(\delta_1)\right]^{(p)}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2!b_3!...b_{m-2n-l-p+2}!}$$
$$+\sum_{k=1}^{\infty}M_k\sum_{l=0}^{m-2n}\delta_1^{k-l}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2!b_3!...b_{m-2n-l+2}!}=0$$
(4)

and

$$
\binom{m-2n}{l}\left[\frac{\partial}{\partial r}\Phi(-n,1;-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2b_3!...b_{m-2n-l+2}!}
$$

$$
\cdot\sum_{p=0}^{m-2n-l}\left[\ln(\delta_1)\right]^{(p)}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2b_3!...b_{m-2n-l-p+2}!}
$$

$$
+\sum_{l=0}^{m-2n}
$$

$$
\binom{m-2n}{l}\left[\Phi(-n,1;-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2b_3!...b_{m-2n-l+2}!}
$$

$$
\cdot\sum_{p=0}^{m-2n-l}(-1)^p\frac{1}{\delta_1^{p+1}}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2b_3!...b_{m-2n-l-p+2}!}
$$

$$
+\sum_{k=1}^{\infty}M_k\sum_{l=0}^{m-2n}
$$

$$
\binom{m-2n}{l}\delta_1^{k-l}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2b_3!...b_{m-2n-l+2}!}
$$

$$
\cdot\sum_{p=0}^{m-2n-l}\beta_1^{(p)}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2b_3!...b_{m-2n-l-p+2}!}\Bigg]
$$

$$
=-\lambda_2\left[\sum_{n=0}^{m}C_n(4a_2^2)n\frac{m!}{(m-2n)!}\sum_{l=0}^{m-2n}\left[\frac{\partial}{\partial r}L_n(-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\beta_2^{b_2}\beta_3^{b_3}...\beta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2b_3!...b_{m-2n-l+2}!}\right]+L\gamma m!\alpha_m,
$$
(4)

where $\beta(\tau)=\dfrac{1}{2a_1^2}\left(\sum_{n=1}^{\infty}\alpha_n\tau^{n-1}\right)^2$. From recurrent expression (4) we express coefficient A_n and making substitution to (4) we can determine B_n as free boundary is given and coefficient C_n can be founded from (4). Then from condition (4) we can describe the recurrent formula for heat flux

$$
\binom{m-2n}{l}\left[\frac{\partial}{\partial r}\Phi(-n,1;-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2b_3!...b_{m-2n-l+2}!}
$$

$$
\cdot\sum_{p=0}^{m-2n-l}\left[\ln(\delta_1)\right]^{(p)}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2b_3!...b_{m-2n-l-p+2}!}
$$

$$
+\sum_{l=0}^{m-2n}
$$

$$
\binom{m-2n}{l}\left[\Phi(-n,1;-\delta_1)\right]^{(l)}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2b_3!...b_{m-2n-l+2}!}
$$

$$
\cdot\sum_{p=0}^{m-2n-l}(-1)^p\frac{1}{\delta_1^{p+1}}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2b_3!...b_{m-2n-l-p+2}!}
$$

$$
+\sum_{k=1}^{\infty}M_k\sum_{l=0}^{m-2n}
$$

$$\binom{m-2n}{l}\delta_1^{k-l}\sum_{b_i}\frac{(m-2n)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l+2}^{b_{m-2n-l+2}}}{b_2b_3!...b_{m-2n-l+2}!}$$
$$\cdot\sum_{p=0}^{m-2n-l}\beta_1^{(p)}\sum_{b_i}\frac{(m-2n-l)!\delta_2^{b_2}\delta_3^{b_3}...\delta_{m-2n-l-p+2}^{b_{m-2n-l-p+2}}}{b_2b_3!...b_{m-2n-l-p+2}!}\Bigg].$$
(4)

Another way, by making substitution the condition (4) to Stefan condition (4) and using temperature $\theta_2(r,t)$ we can find heat flux coefficients.

Numerical test problem

In this section, collocation methods which practical for engineers are shown by taking initial five points $t_i = \frac{2i}{10}$ where $i = 0, 1, 2, 3, 4, 5$. Solution is found exactly and approximately. Effectiveness of these methods considered by taking the following problem

$$\frac{\partial\theta}{\partial r} = \frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}, \quad \alpha(t) < r < \infty, \quad 0 < t < T,$$
(4)

$$\theta(0,0) = 0, \quad \alpha(0) = 0,$$
(4)

$$\theta(r,0) = f(r),$$
(4)

$$-\lambda\frac{\partial\theta(\alpha(t),t)}{\partial r} = P(t) - 2L\gamma\frac{d\alpha}{dt},$$
(4)

$$\theta(\infty,t) = 0.$$
(4)

We represent solution in the form of series for special functions

$$\theta(r,t) = \sum_{n=0}^{\infty} B_n(4t^2)R_{2n,1}(r,t) = \sum_{n=0}^{\infty} A_n(4t^n)L_n\left(-\frac{r^2}{4t}\right) = \sum_{n=0}^{\infty} A_n n! \sum_{k=0}^{n} \frac{2^{2k}r^{2(n-k)}}{k![(n-k)!]^2}t^k,$$

(4)

where $L_n(x)$ are associated Laguerre polynomials.

Exact and approximate solution

If we take free boundary $\alpha(t) = \alpha_0\sqrt{t}$ and function $f(r) = r$, where $\alpha_0 = L = \gamma = \lambda = 1$, then we have the coefficients of $\theta(r,t)$ as follows $B_0 = 1, B_1 = 0.2, B_2 = 0.02, B_3 = 1.41 \times 10^{-3}, B_4 = 7.405 \times 10^{-5}$ an $B_5 = 3.143 \times 10^{-6}$. For heat flux we have $P_1 = -0.4, P_3 = -0.367, P_5 = -0.171$ and $P_0 = P_2 = P_4 = 0$. For approximate solution by using Mathcad 15 calculation, we have for temperature $B_0 = 1, B_1 = 0.2, B_2 = 0.02, B_3 = 1.442\times10^{-3}, B_4 = 6.196\times10^{-5}$ and $B_5 = 5.225\times10^{-6}$ and heat flux $P_1 = -0.4, P_3 = -0.367, P_5 = -0.174$ and $P_0 = P_2 = P_4 = 0$. The Figure 2 shows the exact flux (**P_exact(t)**) and approximate heat flux (**P_app(t)**).

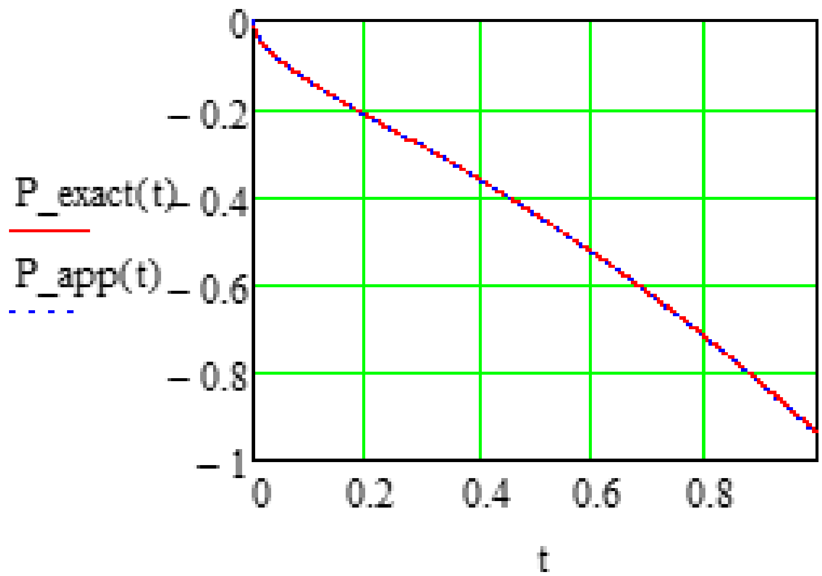

Figure 1: Exact and approximate heat flux functions.

By using formula of relative error we can get Figure **??** which depicts that error estimate is 0.339% near to $t = 1$.

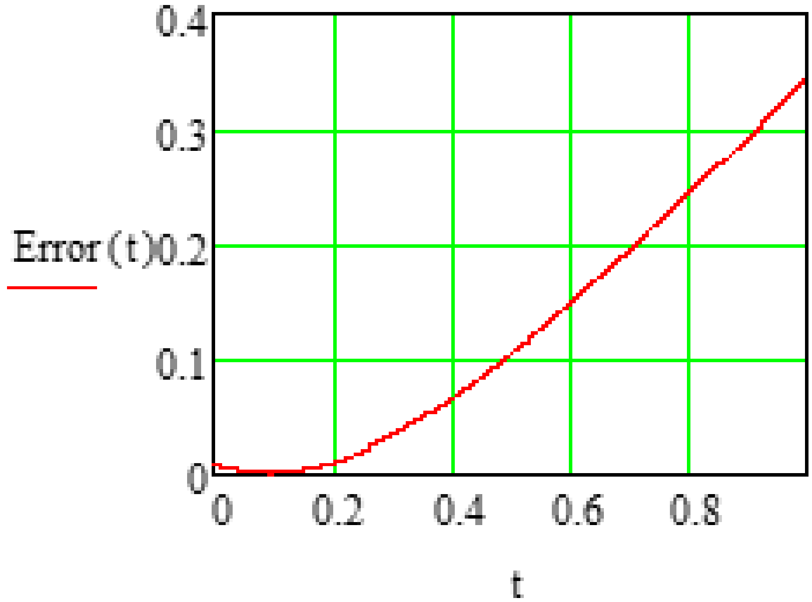

Figure 2: Relative error function between $0 < t < 1$.

Convergence of series

Convergence of series (4)-(4) can be proved as following. Let $\alpha(t_0) = \eta_0$ for any $t = t_0$. Then series (4) can be written as

$$\theta_1(r, t_0) = \sum_{n=0}^{\infty} A_n(4a_1^2 t_0) L_n\left(-\frac{r_0^2}{4a_1^2 t_0}\right) + \sum_{n=0}^{\infty} B_n(4a_1^2 t_0)^n \left[\Phi\left(-n, 1; -\frac{r_0^2}{4a_1^2 t_0}\right) \ln\left(\frac{r_0^2}{4a_1^2 t_0}\right) + \sum_{k=1}^{\infty} M_k \left(-\frac{r_0^2}{4a_1^2 t_0}\right)^k\right].$$

(4)

The series (4) and (4) must be convergence because $\theta_1(\alpha(t), t) = \theta_2(\alpha(t), t) = \theta_m$. Then there exists some constant D_1 independent of n and for the first term of (4) we have

$$|A_n| < D_1/(4a_1^2 t_0)^n L_n\left(-\frac{\eta_0^2}{4a_1^2 t_0}\right).$$

(4)

Since A_n bounded, then multiply both sides of (4) by $(4a_1^2 t)^n L_n\left(-\frac{(\alpha(t))^2}{4a_1^2 t}\right)$ we obtain

$$\sum_{n=0}^{\infty} A_n (4a_1^2 t)^n L_n\Big(-\frac{(\alpha(t))^2}{4a_1^2 t}\Big) < D_1 \sum_{n=0}^{\infty} \frac{(4a_1^2 t)^n L_n\Big(-\frac{(\alpha(t))^2}{4a_1^2 t}\Big)}{(4a_1^2 t_0)^n L_n\Big(-\frac{\eta^2}{4a_1^2 t_0}\Big)} < D_1 \sum_{n=0}^{\infty} \Big(\frac{t}{t_0}\Big)^n$$

(4)

For second term of (4) we consider that there exists some constant D_2 and we obtain

$$|B_n| > D_2 / \Big\{ (4a_1^2 t_0)^n \Big[\Phi\Big(-n, 1; -\frac{r_0^2}{4a_1^2 t_0}\Big) \ln\Big(\frac{r_0^2}{4a_1^2 t_0}\Big) + \sum_{k=1}^{\infty} M_k \Big(-\frac{r_0^2}{4a_1^2 t_0}\Big)^k \Big] \Big\}.$$

(4)

As B_n is bounded, multiplying both sides of (4) by

$$(4a_1^2 t)^n \Big[\Phi\Big(-n, 1; -\frac{(\alpha(t))^2}{4a_1^2 t}\Big) \ln\Big(\frac{(\alpha(t))^2}{4a_1^2 t}\Big) + \sum_{k=1}^{\infty} M_k \Big(-\frac{(\alpha(t))^2}{4a_1^2 t}\Big)^k \Big]$$

we get

(4)

These geometric series and θ_1 convergence for all $r < \mu_0$ and the same θ_2 convergence for all $r > \mu_0$ and $t < t_0$. Convergence for equation (4) and $\alpha(t)$ can be determined analogously.

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