On The Theoretical and Mathematical Analysis of Hydrodynamics Boundary Layer Fluid Flows Regimes

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Abstract

Aerodynamics is the top-level of fluid mechanics science which is deals with the flow of air over bodies like aircraft or any other solid surface. Boundary layer can be classified into hydrodynamics and thermal boundary layer and had wide range of applications in the Aerodynamics. The present work examines the hydrodynamics boundary layer theory over a flat plate. It is interested subject due to its wide range of applications in industry and nature like airplane, missiles, rocket etc. the three laws of physics (mass, momentum and energy) had been derived and then the governing equations of boundary layer had been derivded. The boundary layer is sub-divided into three regions or zones, which they are the laminar, transition and turbulent boundary layer. The Integral Momentum Von-Karman Equation is derived in-full details over the flat plate and then it is used to derived the main parameters in the laminar and turbulent regions like rate of growth of each layer, skin friction coefficient, shear stress and drag coefficient. It is worthy to mention that despite various velocity profiles for laminar region, there is only one profile for the turbulent region which is the seventh root law that suggested by Prandtle. Also, for laminar boundary layer, Blasius proposed a solution that can be used to obtain the drag. The transition zone us discussed also and it is worthy to mention that the analysis of this region is limited in the textbooks of fluid flows and heat transfer.

Nomenclature	Nomenclature
Symbol	**Description**
τ_w	Wall – Shear stress Eq. (1.1)
μ	Dynamics viscosity Eq. (1.1)
$\frac{\mathrm{du}}{\mathrm{dy}}$	Velocity gradient in y direction Eq. (1.1)
ρ	Density Eq. (1.2)
U_∞	Free stream velocity
$\mathrm{Re}_{\mathrm{cr}}$	Critical Reynolds number
x	x – direction
y	y – direction
z	z – direction
u	Component of velocity in x – direction
v	Component of velocity in y – direction
w	Component of velocity in z – direction
P	Pressure
F	Force
a	Acceleration
a_x	Component of acceleration in x – direction Eq. (3.19a)
a_y	Component of acceleration in y – direction Eq. (3.19a)

Nomenclature	Nomenclature
a_z	Component of acceleration in z – direction Eq. (3.19a)
W_x	The work in the x – direction Eq. (3.41)
W_y	The work in the y – direction Eq. (3.42)
W_z	The work in the z – direction Eq. (3.43)
U	Instantaneous velocity Eq. [4.1]
$u'(t)$	fluctuating velocity
u	time average velocity Eq. [4.2]
$\dot{m}$	Mass flow rate
δ^*	Displacement thickness Eq. (5.5)
θ	Momentum thickness Eq. (5.7)
δ^{**}	Kinetic energy thickness Eq. (5.8)
y	Dimensionless length of boundary layer development
U	Dimensionless velocity of boundary layer development
C_f	Skin friction coefficient Eq. (7.17)
D_o	dimensionless parameter of velocity profile Eq. (7.18)
δ	Boundary layer thickness
F_D	Drag force
C_D	Drag coefficient Eq. (7.22)
Re_x	Local Reynolds number
Re_δ	Local Reynolds number at the boundary layer thickness Eq. (7.28)
$\mathrm{Re}_{L,t}$	Local Reynolds number of laminar layer at the transition zone Eq. (7.46)
$\delta_{L,t}$	Laminar boundary layer thickness at the transition region
$\delta_{T,t}$	Turbulent boundary layer thickness at the transition region
$x_{T,t}$	Turbulent length in the transition region

Learning Objective

After completion the reading of this chapter, you should be able to

1. Define the boundary layer theory and its three regions (laminar, transition and turbulent) and the critical Reynolds number for internal and external flow.
2. Derivation the fluid flow and heat transfer laws of physics (mass, momentum and energy) in full – details
3. Definition the turbulence and the different between Navier – Stokes equations and the Reynolds equations.
4. Derivation the boundary layer thickness, momentum thickness and energy thickness
5. Knowledge of the hydrodynamics boundary layer governing equation.
6. Derivation the Momentum Integral Equation for laminar and turbulent regions.
7. Starting from the Momentum Integral Equation to find an expressions for the rate of growth of boundary layer thickness for laminar, transition and turbulent regions.
8. Derivation an expression of drag coefficient in terms of Reynolds number for each boundary layer thickness
9. Recognize the laminar and turbulent velocity profile

Introduction

The boundary layer theory is an interesting subject for the researchers among the world due to its wide range of applications in aerospace engineering like aerodynamics, flows over aircrafts like missiles, airplane, road vehicles and ships. One of the crucial applications of boundary layer is the determination of the drag coefficient of flat plate at zero incidences, flows over airfoil, ships, road vehicles and aircraft. The calculation of the drag is very important as it effects on the fuel consumptions and stability of the body. These days, the fuel resources are decreases and its prices goes up and the fuel consumption is highly influenced by the boundary layer over the road vehicles and aircrafts [1]. Also, one of the problems from the boundary layer is the separation and the stall phenomenon and for this reason there are many aerodynamics modifications to control or delay the separation as it leads to more fuel consumptions. Finally, there are applications in heat transfer between the fluid and the body as in the combustion chamber of spark ignition engines. The flow over a thin flat plate is the first case study of the boundary layer equations of Prandtl (4 February 1875 – 15 August 1953) solved later exactly by Blasius (9 August 1883 – 24 April 1970) in his PhD dissertation on 1908.

When a fluid flows past a solid surface, the velocity of the fluid at that solid surface must be the same as that of the solid surface. If the solid surface is stationary, then the fluid velocity at the surface is zero. So that there is a region close to the surface where the velocity increases from zero at the solid surface to the mean stream velocity (U_∞). In this way, the boundary layer is a narrow region near the solid surface over which both velocity gradient and shear stress are large. It is also known as shear layer theory. The boundary layer theory can be divided into two main types which they are hydrodynamics and thermal layers. The present work illustrates the hydrodynamics boundary layer. The hydrodynamics boundary layer can be divided into two three region or zones, laminar, transition and turbulent as indicated in Figure 1. The well – known Reynolds number is used to distinguish between each layer. For this reason before discussing the hydrodynamics boundary layer it is required to write a section illustrates the concepts of turbulence, Reynolds number and the three laws of physics (mass, energy and momentum of fluid). Form the first look on the schematic diagram it can be noted that the laminar flow is parallel and the fluid flows in on layers gliding smoothly on the adjacent layers. The viscous forces are higher than the inertia forces which makes the laminar flow with small Reynolds number and thus there is no tendencies towards turbulence, eddies formations and instabilities. Beside that the velocity profile is parabolic. So as the flow moves further, there will be eddies formation with higher increasing in Reynolds number which it an indicator on the turbulence had been begun. The velocity profile in turbulent flow regime is logarithmic.

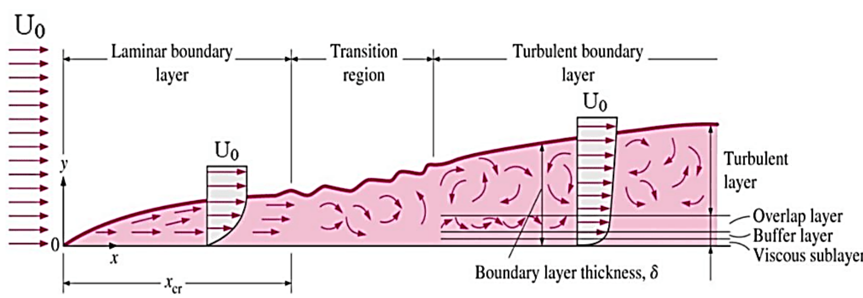


Figure 1 Development of the hydrodynamics boundary layer over a flat plate [2]

Boundary layer theory is a subject connected with the study of velocity gradient, shear stress, forces and energy loss in the boundary layer. For laminar flow, the shear stress can be calculated from Newton's law:

$$\tau_w = \mu \frac{\mathrm{du}}{\mathrm{dy}} \tag{1.1}$$

While for turbulent flow, the shear stress can be obtained from the equation inserted below:

$$\tau_w = 0.0233 \rho U^{\frac{7}{4}} \left(\frac{\nu}{y}\right)^{0.25} \tag{1.2}$$

Reynolds number

The Reynolds number is a criterion which defines the nature of flow if it is laminar, transitional or turbulent by measuring its inertial and viscous forces are given by the equation inserted below[1, 3, 4]:

$$Re = \frac{\rho \ U_\infty \ x}{\mu} \tag{2.1}$$

Where Re is the Reynolds number; ρ is the density of air; U_∞ is the free stream velocity; x is the typical length scale of the system; μ is the dynamics viscosity

As illustrated before, the Reynolds number is used to categorize the nature of the flow type in three regions as illustrated below in Figure 2.

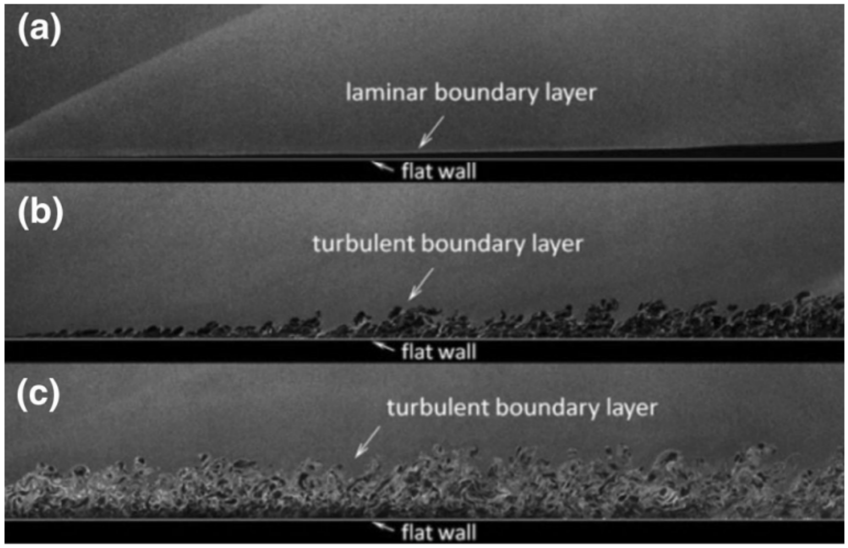


Figure 2 Fluid flow visualization over a flat plate [4]

It is found experimentally, that the turbulent flow occurs at Reynolds number more than the critical Reynolds number as tableted in Table 1

Table 1 Critical Reynolds number for internal and external fluid flows

Type of Flows	**Type of Flows**	**Type of Flows**
External fluid flow Along surface	**External fluid flow Around an obstacle**	**Internal fluid flow**
$Re_{cr} \geq \mathbf{5e+5}$	$Re_{cr} \geq \mathbf{2e+4}$	$Re_{cr} \geq \mathbf{2300}$

Governing Equations

The governing equations of fluid dynamics and heat transfer are the mathematical and physical statements of the three famous conservation laws of physics:

1. "The mass of a fluid is conserved"
2. "The rate of change of momentum equals the sum of the forces on a fluid particle (Newton's second law)"
3. "The rate of change of energy is equal to the sum of the rate of heat addition to and the rate of work done on a fluid particle (first law of thermodynamics)"

The behaviors of fluid will be described in terms of macroscopic properties such as temperature, pressure, velocity and density of fluid as well as space and time derivatives. The fluid element for derivation the conservation laws of physics will be in a 3 dimensional Cartesian coordinates as explained in Figure 3 that inserted below;

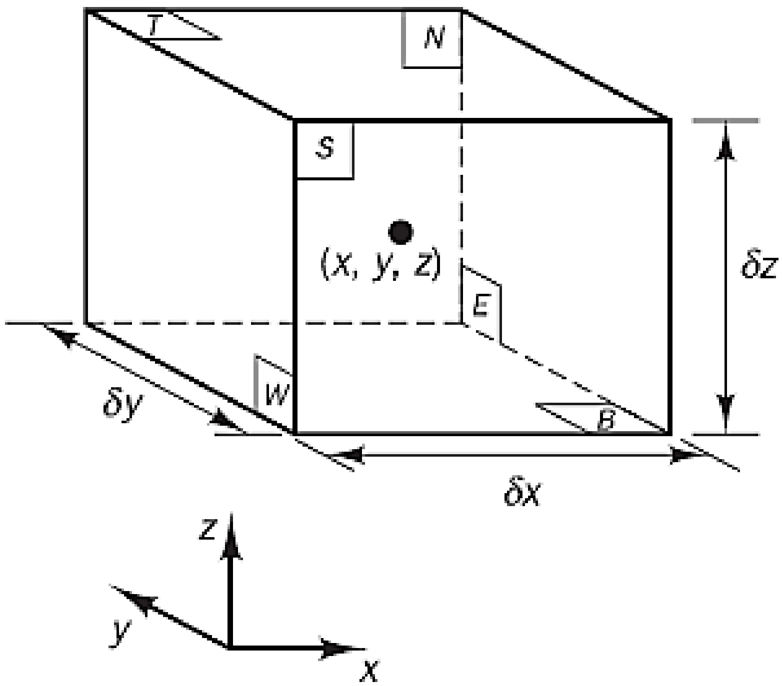


Figure 3 Schematic representation of fluid element [5].

P.S. The fluid thermophysical properties will be as a function of space and time and they write as$\rho = \rho\left(x,y,z,t\right), T = T\left(x,y,z,t\right), P = P\left(x,y,z,t\right), u = u\left(x,y,z,t\right)$for density, temperature, pressure and velocity of fluid.

P.S. Taylors series forward and backward will be used up to first two terms measure from the central point. For example;

forward Taylors series:$u\left(x+x\right) = u\left(x\right) +$ $\underline{\text{δυ}}$

greekxx (3.1)

backward Taylors series

$u\left(x-x\right) = u\left(x\right) -$ $\underline{\text{δυ}}$

greekxx (3.2)

In this way, if we select the pressure for example;

forward Taylors series

$P_2 = P +$ $\underline{\text{δΠ}}$

greekxδξ

ενγλιση2 (3.3)

βαςκωαρδ Ταψλορς σεριες

$P_2 = P - \frac{\delta\Pi}{}$

γρεεκχδξ

ενγλιση2 (3.4)

ὂνσερατιον οφ Μασς [5]

$\frac{\Delta \mathrm{M}_{\sigma\psi\varsigma}}{\Delta\tau} = 0$ (3.5)

$\frac{\partial}{\partial t}\int_{\varsigma} \rho\delta V + \int_{\mathrm{cs}} \rho V \bullet \mathbf{n}\ \mathrm{dA} = 0$ (3.6)

Ρατε οφ ινςρεασες οφ μασς ιν φλυιδ ελεμεντ ςαν βε ςαλςυλατεδ φρομ τηε εχυατιον ινσερτεδ βελοω·

$\frac{\partial}{\partial t}\int_{\varsigma} \rho\delta V = \frac{\partial\rho}{\partial t}\delta\xi\delta\psi\delta\zeta$ (3.7)

Ρεγαρδινγ τηε νετ ρατε οφ μασς φλοω ρατε τηρουγη εαςη φαςε οφ τηε φλυιδ ελεμεντ ωηιςη ις τηε δενσιτψ μυλτιπλιεδ βψ τηε ελοςιτψ ςομπονεντς περπενδιςυλαρ το τηε φαςε βψ τηε ςροσς σεςτιοναλ αρεα ας ιλλυστρατεδ ιν Φιγυρε 4. Τηε αναλψσις ις ινσερτεδ βελοω·

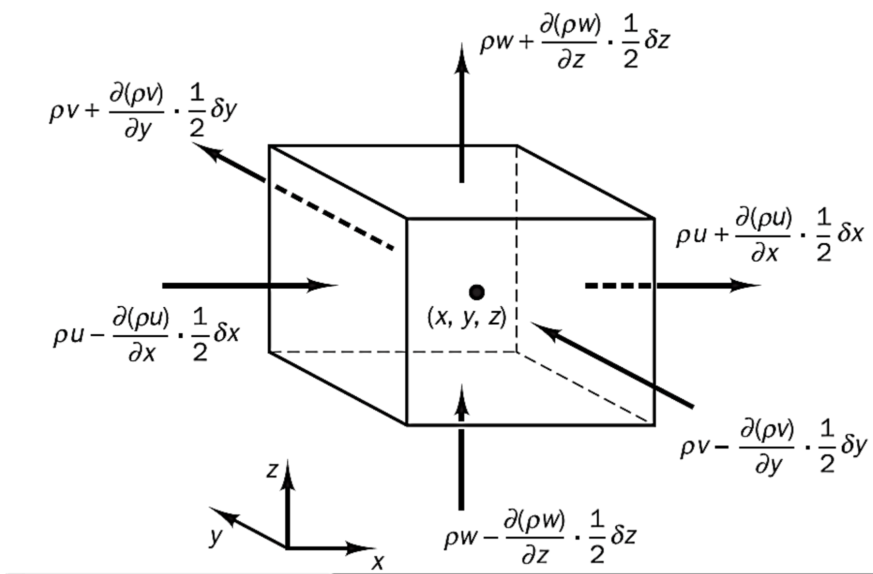


Figure 4 Mass in and out of the box of the fluid [5]

Now let "ρu" is the mass flow rate per unit area at the center of the element, then with help of forward finite difference scheme along the x – direction

Forward FDM along the right surface

$\frac{\partial \rho u}{\partial x} = \frac{\rho\upsilon]_{x+\delta x} - \rho\upsilon]_{\varsigma\epsilon\nu\tau\epsilon\rho}}{}$

english$\delta x/2$

$\rho\upsilon]_{x+\delta x} = \rho\upsilon]_{\mathrm{center}} + \frac{\partial \rho u}{\partial x}\frac{\delta\xi}{}$

english2

Backward FDM along the left surface

$\frac{\partial \rho u}{\partial x} = \frac{\rho\upsilon]_{\varsigma\epsilon\nu\tau\epsilon\rho} - \rho\upsilon]_{x-\delta x}}{}$

english$\delta x/2$

$\rho\upsilon\rceil_{x-\delta x} = \rho\upsilon\rceil_{\text{center}} - \frac{\partial \rho u}{\partial x}\underline{\delta\xi}$

english2

Net rate of mass in x – direction $= (\rho u\rceil_{x+\delta x} - \rho\upsilon\rceil_{x-\delta x})\delta y\delta z$

$= \Big\{ \rho\upsilon\rceil_{\text{center}} + \frac{\partial \rho u}{\partial x}\underline{\delta\xi}$

english2-$\Big(\rho\upsilon\rceil_{\text{center}} - \frac{\partial \rho u}{\partial x}\underline{\delta\xi}$

english2δψδζ

$= \Big\{ \rho\upsilon\rceil_{\text{center}} + \frac{\partial \rho u}{\partial x}\underline{\delta\xi}$

english2- $\rho\upsilon\rceil_{\text{center}} + \frac{\partial \rho u}{\partial x}\underline{\delta\xi}$

english2δψδζ

$$\textit{Net rate of mass in x direction} = \frac{\partial \rho u}{\partial x}\delta\xi\delta\psi\delta\zeta \tag{3.8}$$

Similarly at y – direction

$\frac{\partial \rho v}{\partial y} = \frac{\rho\rceil_{y+\delta y} - \rho\rceil_{\varsigma\epsilon\nu\tau\epsilon\rho}}{}$

english$\delta y/2$Forward FDM along the right surface

$\rho\rceil_{y+\delta y} = \rho\rceil_{\text{center}} + \frac{\partial \rho v}{\partial y}\underline{\delta\psi}$

english2

$\frac{\partial \rho v}{\partial y} = \frac{\rho\rceil_{\varsigma\epsilon\nu\tau\epsilon\rho} - \rho\rceil_{y-\delta y}}{}$

english$\delta y/2$Backward FDM along the left surface

$\rho\rceil_{y-\delta y} = \rho\rceil_{\text{center}} - \frac{\partial \rho v}{\partial y}\underline{\delta\psi}$

english2

Net rate of mass in x – direction $= (\rho v\rceil_{x+\delta x} - \rho\rceil_{y-\delta y})\delta x\delta z$

$= \Big\{ \rho\rceil_{\text{center}} + \frac{\partial \rho v}{\partial y}\underline{\delta\psi}$

english2-$\Big(\rho\rceil_{\text{center}} - \frac{\partial \rho v}{\partial y}\underline{\delta\psi}$

english2δξδζ

$= \Big\{ \rho\rceil_{\text{center}} + \frac{\partial \rho v}{\partial y}\underline{\delta\psi}$

english2- $\rho\rceil_{\text{center}} + \frac{\partial \rho v}{\partial y}\underline{\delta\psi}$

english2δξδζ

$$\textit{Net rate of mass in y direction} = \frac{\partial \rho v}{\partial y}\delta\xi\delta\psi\delta\zeta \tag{3.9}$$

Similarly at z – direction

$\frac{\partial \rho w}{\partial z} = \frac{\rho\omega\rceil_{z+\delta z} - \rho\omega\rceil_{\varsigma\epsilon\nu\tau\epsilon\rho}}{}$

english$\delta z/2$Forward FDM along the right surface

$\rho\omega\rceil_{z+\delta z} = \rho\omega\rceil_{\text{center}} + \frac{\partial \rho w}{\partial z}\underline{\delta\zeta}$

english2

$\frac{\partial \rho z}{\partial z} = \frac{\rho\omega]_{\varsigma\varepsilon\nu\tau\varepsilon\rho} - \rho\omega]_{z-\delta z}}{}$

english$\delta z/2$Backward FDM along the left surface

$\rho\omega]_{z-\delta z} = \rho\omega]_{\text{center}} - \frac{\partial \rho w}{\partial z}\frac{\delta\zeta}{}$

english2

Net rate of mass in z – direction $= (\rho w]_{z+\delta z} - \rho\omega]_{z-\delta z})\delta x\delta y$

$= \Big\{ \rho\omega]_{\text{center}} + \frac{\partial \rho w}{\partial z}\frac{\delta\zeta}{}$

english2-$\Big(\rho\omega]_{\text{center}} - \frac{\partial \rho w}{\partial z}\frac{\delta\zeta}{}$

english2δξδψ

$= \Big\{ \rho\omega]_{\text{center}} + \frac{\partial \rho w}{\partial z}\frac{\delta\zeta}{}$

english2- $\rho\omega]_{\text{center}} + \frac{\partial \rho w}{\partial z}\frac{\delta\zeta}{}$

english2δξδψ

$$Net\ rate\ of\ mass\ in\ z\ \ direction = \frac{\partial \rho w}{\partial z}\delta\xi\delta\psi\delta\zeta \tag{3.10}$$

Thus;

Net flow rate of mass flow rate $= \left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right]\delta\xi\delta\psi\delta\zeta$

$\frac{\partial \rho}{\partial t}\delta\xi\delta\psi\delta\zeta + \left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right]\delta x\delta y\delta z = 0$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \tag{3.11}$$

The above equation represents the continuity equation which is one of the fundamental equations of fluid mechanics. It is valid for transient, compressible or incompressible fluid flow. It can be written in vector form as follow;

$$\frac{\partial \rho}{\partial t} + \nabla \bullet \rho\mathbf{V} = 0 \tag{3.12}$$

Two special cases that the researchers interested on them due to wide range of their applications in engineering and industry like solar collectors, internal pipe flow, nanofluid enclosure along with natural convection which they are;

1. For steady, compressible fluid flow whick makes the density as a function of space only;$\nabla \bullet \rho\mathbf{V} = 0$ (3.13)
2. For incompressible flow;$\nabla \bullet \mathbf{V} = 0$ (3.14)$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (3.15)
3. **Momentum Equation**

Before derivatives the momentum and energy equations, it is very important to full – understanding the following concept which is called Material Derivative.

Physically, any property is a function of space and time. The space is represented by the coordinates (x, y, z) and time which is denoted as t. so that we can write the total or substantive or material derivative which is denoted as ∅ as indicated below;

$$\frac{D\varnothing}{\mathrm{Dt}} = \frac{\partial\varnothing}{\partial t} + \frac{\partial\varnothing}{\partial x}\frac{\mathrm{dx}}{\mathrm{dt}} + \frac{\partial\varnothing}{\partial y}\frac{\mathrm{dy}}{\mathrm{dt}} + \frac{\partial\varnothing}{\partial z}\frac{\mathrm{dz}}{\mathrm{dt}} \tag{3.16}$$

$\frac{\mathrm{dx}}{\mathrm{dt}} = u, \qquad \frac{\mathrm{dy}}{\mathrm{dt}} = v, \qquad \frac{\mathrm{dz}}{\mathrm{dt}} = w$

$$\frac{D\varnothing}{\mathrm{Dt}} = \frac{\partial\varnothing}{\partial t} + u\frac{\partial\varnothing}{\partial x} + v\frac{\partial\varnothing}{\partial y} + w\frac{\partial\varnothing}{\partial z} \tag{3.17}$$

So that the acceleration can be written as below;

$$a = \frac{D\mathbf{V}}{\mathrm{Dt}} = \frac{\partial\mathbf{V}}{\partial t} + u\frac{\partial\mathbf{V}}{\partial x} + v\frac{\partial\mathbf{V}}{\partial y} + w\frac{\partial\mathbf{V}}{\partial z} \tag{3.18}$$

A shorthand notation for the material derivatives operator is;

$$\frac{D\varnothing}{\mathrm{Dt}} = \frac{\partial\varnothing}{\partial t} + (\mathbf{V}\bullet\nabla) \tag{3.19}$$

Where the $\mathbf{V}$ is the velocity vector and it is given by=$\mathbf{u}\ \hat{\mathrm{i}} + \mathbf{v}\ \hat{\mathrm{j}} + \mathbf{w}\ \hat{\mathrm{k}}$, the velocity gradient is denoted as ∇ and it is given by$\nabla\varnothing = \frac{\partial\varnothing}{\partial\mathbf{x}}\ \hat{\mathrm{i}} + \frac{\partial\varnothing}{\partial\mathbf{y}}\ \hat{\mathrm{j}} + \frac{\partial\varnothing}{\partial\mathbf{z}}\ \hat{\mathrm{k}}$

As an example; acceleration – components will be;

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{3.19a}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{3.19b}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{3.19c}$$

Newton's law physically states that "the rate of change of momentum of a fluid particle equals to the sum of forces on the particle". Mathematically, it could be written as inserted below

$$\sum F = m\ a \tag{3.20}$$

There are two major forces acting on fluid particle;

Surface forces

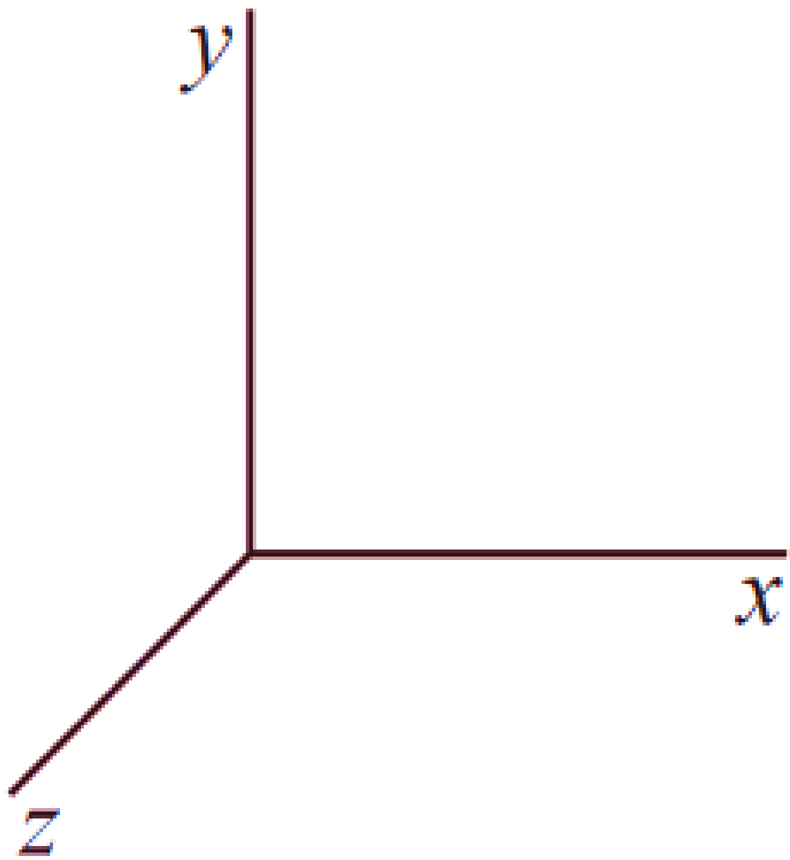
- Pressure forces
- Viscous forces

Body forces

- Gravity force
- Centrifugal force
- Coriolis force
- Electromagnetic field

It is commonly on CFD problems related to heat transfer and fluid mechanics to include two forces which is a surface force as a separated force and the body force as a source term.

Now let us analysis the surface forces or stresses which contains normal stress and shear stress. The shear stress is nothing but the force magnitude divided by the area. The stress opposite in the direction to the proposed direction as shown in Figure 5.



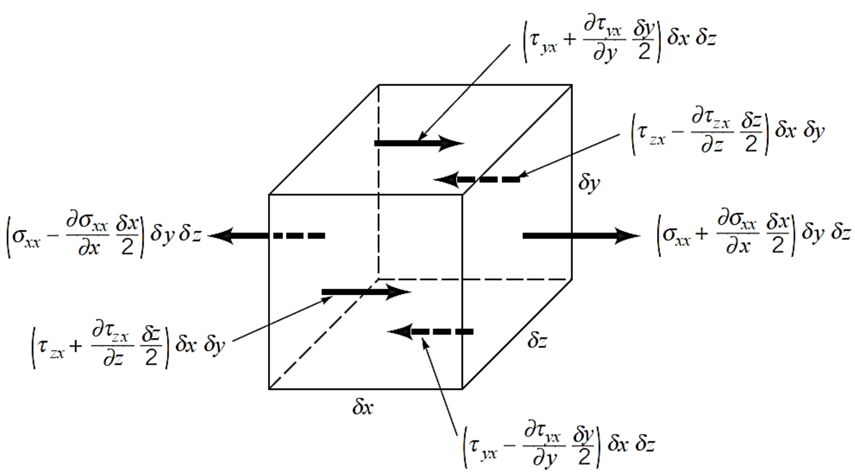


Figure 5 fluid element with the normal and shear stress applied on it [6]

Net force in the x – direction in the right and left face is

$\left[\sigma_{\mathrm{xx}} + \frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\frac{\delta\xi}{}\right.$

english2-$\left(\sigma_{\mathrm{xx}} - \frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\frac{\delta\xi}{}\right.$

english2δψδζ $= \frac{\partial \sigma_{\mathrm{xx}}}{\partial x}$ δξδψδζ (3.21)

Net force in the x – direction in the top and bottom face is

$\Big[\tau_{\mathrm{yx}} + \frac{\partial \tau_{\mathrm{yx}}}{\partial y}\frac{\delta\psi}{}$

english2-$\Big(\tau_{\mathrm{yx}} - \frac{\partial \tau_{\mathrm{yx}}}{\partial y}\frac{\delta\psi}{}$

english2δψδζ $= \frac{\partial \tau_{\mathrm{yx}}}{\partial x}$ δξδψδζ (3.22)

Net force in the x – direction in the front and back face is

$\Big[\tau_{\mathrm{zx}} + \frac{\partial \tau_{\mathrm{zx}}}{\partial z}\frac{\delta\zeta}{}$

english2-$\Big(\tau_{\mathrm{zx}} - \frac{\partial \tau_{\mathrm{zx}}}{\partial z}\frac{\delta\zeta}{}$

english2δψδζ $= \frac{\partial \tau_{\mathrm{zx}}}{\partial z}$ δξδψδζ (3.23)

$$\delta\Phi_s = \left(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x} + \frac{\partial \tau_{\mathrm{yx}}}{\partial y} + \frac{\partial \tau_{\mathrm{zx}}}{\partial z}\right)\delta\xi\delta\psi\delta\zeta \quad (3.24)$$

Now it is the time to write down the first form of the equation of motion in the x – direction by conjunction the body and surface forces

$\sum F = \delta m\ a_x\ ,\sum F = \delta\Phi_s + S_M\quad ,\delta m = \rho\delta x\delta y\delta z\ ,\ S_M = \rho g$

$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

$\sum F = \delta m\ a_x$

$$\left(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x} + \frac{\partial \tau_{\mathrm{yx}}}{\partial y} + \frac{\partial \tau_{\mathrm{zx}}}{\partial z}\right)\delta\xi\delta\psi\delta\zeta + S_M = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\delta\xi\delta\psi\delta\zeta$$

By canceling δxδyδz and substitute the source term$S_M = \rho g_x$

$$\left(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x} + \frac{\partial \tau_{\mathrm{yx}}}{\partial y} + \frac{\partial \tau_{\mathrm{zx}}}{\partial z}\right) + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Then, we can obtain the x – component of momentum equation

$$\left(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x} + \frac{\partial \tau_{\mathrm{yx}}}{\partial y} + \frac{\partial \tau_{\mathrm{zx}}}{\partial z}\right) + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \quad (3.25)$$

Similarly we can obtain the y – component of momentum equation

$$\left(\frac{\partial \sigma_{\mathrm{yy}}}{\partial x} + \frac{\partial \tau_{\mathrm{xy}}}{\partial y} + \frac{\partial \tau_{\mathrm{zy}}}{\partial z}\right) + \rho g_y = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \quad (3.26)$$

Similarly we can obtain the z – component of momentum equation

$$\left(\frac{\partial \sigma_{\mathrm{zz}}}{\partial x} + \frac{\partial \tau_{\mathrm{xz}}}{\partial y} + \frac{\partial \tau_{\mathrm{yz}}}{\partial z}\right) + \rho g_z = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \quad (3.27)$$

From fluid mechanics textbooks; The normal stresses [6];

$$\sigma_{\mathrm{xx}} = -P + 2\mu\left(\frac{\partial u}{\partial x}\right) \quad (3.28)$$

$$\sigma_{\mathrm{yy}} = -P + 2\mu\left(\frac{\partial v}{\partial y}\right) \quad (3.29)$$

$$\sigma_{\mathrm{zz}} = -P + 2\mu\left(\frac{\partial u}{\partial x}\right) \quad (3.30)$$

The shearing stresses [6];

$$\tau_{\mathrm{xy}} = \tau_{\mathrm{yx}} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad (3.31)$$

$$\tau_{\mathrm{xz}} = \tau_{\mathrm{zx}} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad (3.32)$$

$$\tau_{\mathrm{yz}} = \tau_{\mathrm{zy}} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{3.33}$$

Navier – Stokes Equations

These are one of the hardest PDEs that had never solved exactly and it is one of the one million USD equations. These equations along with the conservation of mass and energy equations are the corner stone of all of the fluid flow and heat transfer problems due to their wide range of applications. As there is not exact solutions, an approximate solutions using CFD had been developed for various problems and using of different models. Since the present work concentrates on the turbulent flow, various turbulence models will be discussed in full – details later.

Let us first obtained the Navier – Stokes Equation in x – direction;

$$\left(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x} + \frac{\partial \tau_{\mathrm{yx}}}{\partial y} + \frac{\partial \tau_{\mathrm{zx}}}{\partial z}\right) + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial}{\partial x}\left(-P + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Now let us develop the N – S equation in x – direction for a special case study which involves for incompressible, Inviscid flow;

$$-\frac{\partial P}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial y}\frac{\partial v}{\partial x} + \mu\frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \mu\frac{\partial}{\partial z}\frac{\partial w}{\partial x} + \mu\frac{\partial}{\partial z}\frac{\partial u}{\partial z} + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Now, since$\mu\frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \mu\frac{\partial}{\partial x}\frac{\partial v}{\partial y}and\mu\frac{\partial}{\partial z}\frac{\partial w}{\partial x} = \mu\frac{\partial}{\partial x}\frac{\partial w}{\partial z}$

and by put$2\mu\frac{\partial^2 u}{\partial x^2} = \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x^2}$

Then,

$$-\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \mu\frac{\partial}{\partial y}\frac{\partial u}{\partial y} + \mu\frac{\partial}{\partial x}\frac{\partial w}{\partial z} + \mu\frac{\partial}{\partial z}\frac{\partial u}{\partial z} + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$-\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \mu\frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial}{\partial x}\frac{\partial w}{\partial z} + \mu\frac{\partial^2 u}{\partial z^2} + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Let us re-arrange them so that we can use continuity equation;

$$-\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial z^2} + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$-\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial z^2} + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$-\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

In this way, the N – S Equation in x – direction will be

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{3.34}$$

Similarly and using of the same procedure, we can obtain the N – S equations in

y – direction

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{3.35}$$

z – direction

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{3.36}$$

Energy Equation

Figure 6 demonstrates the two components of stresses which they are the normal and shear stresses on a fluid particle. The normal stress is denoted as"σ_{xx}" and the shear stress is denoted by the symbol"τ_{xy}".

Firstly, we consider the x – component of the forces due to pressure, normal and shear stresses components as illustrated below in Fig. .

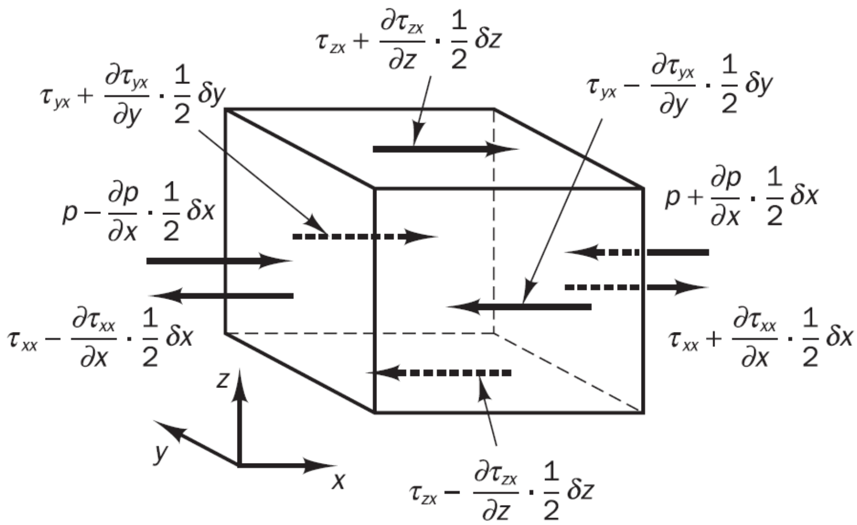


Figure 6 two components of stresses on a fluid particle [5]

Forces on the left and right faces are denoted as F_1

$F_1 = \Big[\Big(P - \frac{\partial P}{\partial x}\,\overline{\delta\xi}$

english2$-\Big(\sigma_{\mathrm{xx}} - \frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\,\overline{\delta\xi}$

english2$\delta y \delta z + \Big[-\Big(P + \frac{\partial P}{\partial x}\,\overline{\delta\xi}$

english2$+\Big(\sigma_{\mathrm{xx}} + \frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\,\overline{\delta\xi}$

english2$\delta\psi\delta\zeta$

$F_1 = \Big[\Big(-\frac{\partial P}{\partial x}\,\overline{\delta\xi}$

english2$+\Big(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\,\overline{\delta\xi}$

english2$\delta y \delta z + \Big[-\Big(\frac{\partial P}{\partial x}\,\overline{\delta\xi}$

english2$+\Big(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\,\overline{\delta\xi}$

english2$\delta\psi\delta\zeta$

$F_1 = \left[\left(-\frac{\partial P}{\partial x}\delta\xi\right) + \left(\frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\delta\xi\right)\right]\delta\psi\delta\zeta$

$$F_1 = \left[-\frac{\partial P}{\partial x} + \frac{\partial \sigma_{\mathrm{xx}}}{\partial x}\right]\delta\xi\delta\psi\delta\zeta \qquad (3.37)$$

Forces on the front and back faces are denoted as F_2

$F_2 = \Big(\tau_{\text{yx}} + \frac{\partial \tau_{\text{yx}}}{\partial y} \frac{\delta\psi}{}$

english$2\delta x\delta y - \Big(\tau_{\text{yx}} - \frac{\partial \tau_{\text{yx}}}{\partial y} \frac{\delta\psi}{}$

english2δξδψ

$$F_2 = \frac{\partial \tau_{\text{yx}}}{\partial y} \delta\xi\delta\psi\delta\psi \qquad (3.38)$$

Forces on the top and bottom faces are denoted as F_3

$F_3 = \Big(\tau_{\text{yx}} + \frac{\partial \tau_{\text{yx}}}{\partial z} \frac{\delta\psi}{}$

english$2\delta x\delta y - \Big(\tau_{\text{yx}} - \frac{\partial \tau_{\text{yx}}}{\partial z} \frac{\delta\psi}{}$

english2δξδψ

$$F_3 = \frac{\partial \tau_{\text{zx}}}{\partial z} \delta\xi\delta\psi\delta\psi \qquad (3.39)$$

Then, the net forces in the x – direction is equal to the sum of all of the three forces;

$$F_x = F_1 + F_2 + F_3 = \Big\{\Big[-\frac{\partial P}{\partial x} + \frac{\partial \sigma_{\text{xx}}}{\partial x}\Big] + \frac{\partial \tau_{\text{yx}}}{\partial y} + \frac{\partial \tau_{\text{zx}}}{\partial z}\Big\} \delta\xi\delta\psi\delta\psi$$

$$F_x = \Big[\frac{\partial(-P+\sigma_{\text{xx}})}{\partial x} + \frac{\partial \tau_{\text{yx}}}{\partial y} + \frac{\partial \tau_{\text{zx}}}{\partial z}\Big] \delta\xi\delta\psi\delta\psi \qquad (3.40)$$

Now, the work done in the x – direction which is denoted as $W_x$will be the product of the force by the velocity;

$$W_x = \Big[\frac{\partial[u(-P+\sigma_{\text{xx}})]}{\partial x} + \frac{\partial[u\tau_{\text{yx}}]}{\partial y} + \frac{\partial[u\tau_{\text{zx}}]}{\partial z}\Big] \delta\xi\delta\psi\delta\psi \qquad (3.41)$$

Similarly, the surface stresses components in the y and z direction is given by the mathematical formula indicated below;

$$W_y = \Big[\frac{\partial[v\tau_{\text{xy}}]}{\partial x} + \frac{\partial[v(-P+\sigma_{\text{yy}})]}{\partial y} + \frac{\partial[v\tau_{\text{zy}}]}{\partial z}\Big] \delta\xi\delta\psi\delta\psi \qquad (3.42)$$

$$W_z = \Big[\frac{\partial[w\tau_{\text{xz}}]}{\partial x} + \frac{\partial[w\tau_{\text{yz}}]}{\partial y} + \frac{\partial[w(-P+\sigma_{\text{zz}})]}{\partial z}\Big] \delta\xi\delta\psi\delta\psi \qquad (3.43)$$

Now, it is the time to collect the terms that contains the pressure together as indicated below;

$$-\frac{\partial[\text{uP}]}{\partial x} - \frac{\partial[\text{vP}]}{\partial x} - \frac{\partial[\text{wP}]}{\partial x} = -div\,(P\mathbf{u})$$

The total work done on the fluid particle will be;

$W = -div\,(P\mathbf{u}) + \Big[\frac{\partial(u\sigma_{\text{xx}})}{\partial x} + \frac{(\upsilon\tau_{\psi\xi})}{}$

english$\partial y + \frac{\partial(\upsilon\tau_{\zeta\xi})}{}$

english$\partial z + \frac{\partial(v\sigma_{\text{yy}})}{\partial x} + \frac{(\tau_{\xi\psi})}{}$

english$\partial y + \frac{\partial(\tau_{\zeta\psi})}{}$

english$\partial z + \frac{\partial(w\sigma_{\text{zz}})}{\partial x} + \frac{(\omega\tau_{\psi\zeta})}{}$

english$\partial y + \frac{\partial(\omega\tau_{\xi\zeta})}{}$

english∂z (3.44)

Now let us find the Energy flux due to conduction heat transfer

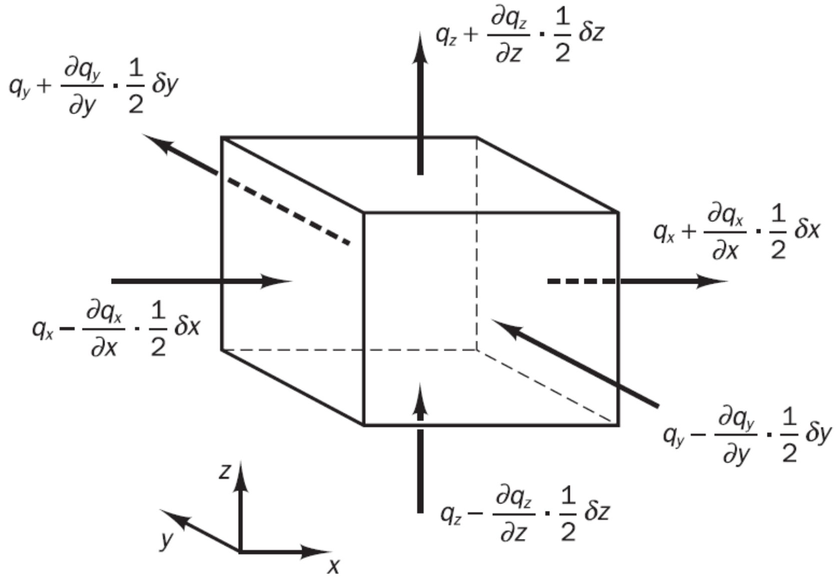


Figure 7 the net heat transfer in a control volume [5]

The net heat transfer in the x – direction is equal to

$\Big[\Big(q_x - \frac{\partial q_x}{\partial x}\frac{\delta\xi}{}$

english2$-\Big(q_x + \frac{\partial q_x}{\partial x}\frac{\delta\xi}{}$

english2$\delta\psi\delta\zeta = -\frac{\partial q_x}{\partial x}\delta\xi\delta\psi\delta\zeta$

The net heat transfer in the y – direction is equal to

$\Big[\Big(q_y - \frac{\partial q_y}{\partial y}\frac{\delta\psi}{}$

english2$-\Big(q_y + \frac{\partial q_y}{\partial y}\frac{\delta\psi}{}$

english2$\delta\xi\delta\zeta = -\frac{\partial q_y}{\partial x}\delta\xi\delta\psi\delta\zeta$

The net heat transfer in the z – direction is equal to

$\Big[\Big(q_z - \frac{\partial q_z}{\partial z}\frac{\delta\zeta}{}$

english2$-\Big(q_z + \frac{\partial q_z}{\partial z}\frac{\delta\zeta}{}$

english2$\delta\xi\delta\psi = -\frac{\partial q_z}{\partial z}\delta\xi\delta\psi\delta\zeta$

Then; the total heat rate per unit volume is the sum of all of the heat flow across the boundaries divided by δxδyδz

$$-\frac{\partial q_x}{\partial x} = -\frac{\partial q_y}{\partial y} = -\frac{\partial q_z}{\partial z} = -div\,(\mathbf{q}) \qquad (3.45)$$

The heat flux and the temperature gradient can related by Fourier's law;

$q_x = -k\frac{\partial T}{\partial x} q_y = -k\frac{\partial T}{\partial y} q_z = -k\frac{\partial T}{\partial z}$

$\mathbf{q} = -k \;\; grad\ T$

$-div\,(\mathbf{q}) = div\,(\mathrm{k\;\; grad\;\; T}\;)$ (3.46)

In this way, Energy Equation can be written as below;

$\rho\frac{\mathrm{DE}}{\mathrm{Dt}} == -div\,(P\mathbf{u}) + \Big[\frac{\partial(u\sigma_{\mathrm{xx}})}{\partial x} + \frac{(\upsilon\tau_{\psi\xi})}{}$

english$\partial y + \frac{\partial(\upsilon\tau_{\zeta\xi})}{}$

english$\partial z + \frac{\partial(v\sigma_{\mathrm{yy}})}{\partial x} + \frac{(\tau_{\xi\psi})}{}$

english$\partial y + \frac{\partial(\tau_{\zeta\psi})}{}$

english$\partial z + \frac{\partial(w\sigma_{\mathrm{zz}})}{\partial x} + \frac{(\omega\tau_{\psi\zeta})}{}$

english$\partial y + \frac{\partial(\omega\tau_{\xi\zeta})}{}$

english$\partial z + div\,(\mathrm{k\;\; grad\;\; T}\;)$ (3.47)

1. **Turbulence**
2. **What is Turbulence?**

Turbulence is a the top level and a leading subject of fluid flow researches and during the last century some of the famous mathematician worked in this specific area like Reynolds, Taylor, Von – Karman, Parbdtl and his PhD student Blasius. Turbulence may be defined as a random, irregular, unpredictable motion in which each quantity of fluid flow properties fluctuates continuously with respect to the time and space [5]. Turbulence leads to increases drag, mixing, energy dissipation and heat transfer beside that it is a 3 – D flow [7]. For example, Figure 8 displays the water jet image visualized using laser-induced fluorescence technique under turbulent flow. It can be seen how the turbulence effect is high on the irregularity of the water distribution. Also, the turbulence is a recommended technique to increases the flame speed which enhance the heat release as illustrated in Figure 9 which is tangential swirl burner. Based on Figure 10 it can be seen that the instantaneous velocity fluctuate about its average value and can be written as indicated below;

$U = u + \acute{u}\,(t)$ (4.1)

Where $\acute{u}\,(t)$ it is the fluctuating velocity

u it is the time average velocity and can be calculated from

$u = \frac{1}{t}\int_0^t \mathrm{U\ dt}$ (4.2)

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Figure 8 An axisymmetric water jet image measured visualized using laser-induced fluorescence technique [8]

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Figure 9 Tangential air insert to generate more turbulence [9]

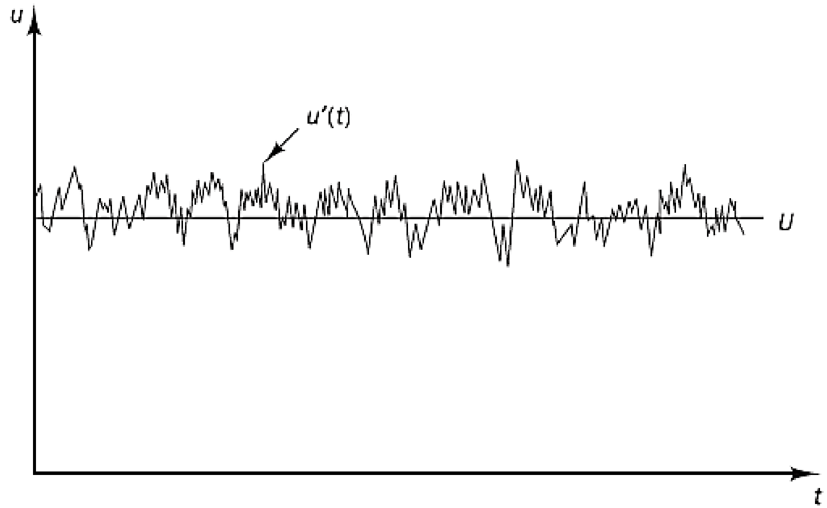

Figure 10 Influence of Turbulence on velocity history [5]

Reynolds equations

In the Navier – Stokes equations, the turbulence motion had been neglected and only the mean viscous stresses and the apparent turbulent stresses had been taken in the considerations. In this way, the laminar and turbulent fluid flows can be treated in a common frame work of the Navier – Stokes equations. Thus, if the turbulences stresses included in the equation of motion, then the resulted equation called the Reynolds equation.

From fluid mechanics textbooks; The normal stresses [6];

$$\sigma_{\mathrm{xx}} = -P + 2\mu\left(\frac{\partial u}{\partial x}\right) - \rho u'^{2} \tag{4.3}$$

$$\sigma_{\mathrm{yy}} = -P + 2\mu\left(\frac{\partial v}{\partial y}\right) - \rho v'^{2} \tag{4.4}$$

$$\sigma_{\mathrm{zz}} = -P + 2\mu\left(\frac{\partial u}{\partial x}\right) - \rho w'^{2} \tag{4.5}$$

The shearing stresses [6];

$$\tau_{\mathrm{xy}} = \tau_{\mathrm{yx}} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) - \rho u'v' \tag{4.6}$$

$$\tau_{\mathrm{xz}} = \tau_{\mathrm{zx}} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - \rho u'w' \tag{4.7}$$

$$\tau_{\mathrm{yz}} = \tau_{\mathrm{zy}} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) - \rho w'v' \tag{4.8}$$

If we substitute the above formulas eq. (4.3) – (4.8) in the Navier – Stokes equation, the Reynolds equations will be as indicated below;

$$\left(\frac{Du}{\mathrm{Dt}}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 u - \left[\frac{\partial}{\partial x}\left(u'^{2}\right) + \frac{\partial}{\partial y}\left(u'v'\right) + \frac{\partial}{\partial z}\left(u'w'\right)\right] \tag{4.9}$$

$$\left(\frac{Dv}{\mathrm{Dt}}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 v - \left[\frac{\partial}{\partial x}\left(\acute{v}\acute{u}\right) + \frac{\partial}{\partial y}\left(\acute{v}^2\right) + \frac{\partial}{\partial z}\left(\acute{v}\acute{w}\right)\right] \tag{4.10}$$

$$\left(\frac{Dw}{\mathrm{Dt}}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 w - \left[\frac{\partial}{\partial x}\left(\acute{w}\acute{u}\right) + \frac{\partial}{\partial y}\left(\acute{w}\acute{v}\right) + \frac{\partial}{\partial z}\left(\acute{w}^2\right)\right] \tag{4.11}$$

In the present work, Navier – Stokes equation will be used instead of Reynolds equations for the boundary layer analysis as it will be demonstrated in the next section.

1. **Estimation of boundary layer characteristics**
2. Displacement thicknessδ^*

It is the distance (y) by which the external free stream is effectively displaced to formulation of boundary layer.

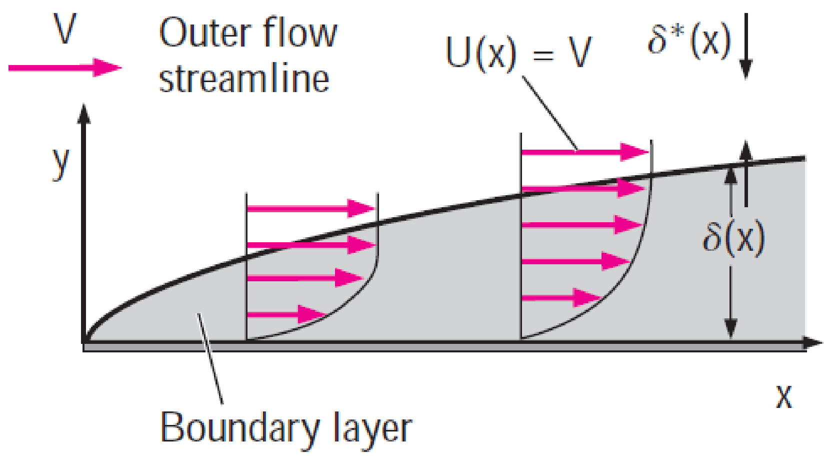

Figure 11 displacement thickness [2]

If a free stream of velocity U_∞ is effectively displaced byδ^*. The loss of the mass flow rate per unit time is given by:-

$$\dot{m} = \rho_\infty U_\infty \delta^* \tag{5.1}$$

The loss of the mass flow rate per unit time is given by:-

$$d\dot{m} = \rho\left(U_\infty - u\right)\mathrm{dy}$$

Then the total mass flow rate per unit time is:

$$\dot{m} = \int_0^\delta \rho\left(U_\infty - u\right)\mathrm{dy} \tag{5.2}$$

By equation the above equations, we get

$$\rho_\infty U_\infty \delta^* = \int_0^\delta \rho\left(U_\infty - u\right)\mathrm{dy}$$

If the fluid flow is assumed to be incompressible i.e, the density remains constant, and then the above equation will be written as follow;

$$\rho_\infty U_\infty \delta^* = \int_0^\delta \rho\left(U_\infty - u\right)\mathrm{dy}$$

$\delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) \mathrm{dy}$ (5.3)

In the fluid mechanics it is recommended to transform the equations into non-dimensional form. In this way;

Let$y = \frac{y}{\delta}$ and $U = \frac{u}{U_\infty}$we will get;

$\delta^* = \int_0^1 \left(1 - \frac{u}{U_\infty}\right) \delta\delta y$

$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - U\right) dy$ (5.4)

momentum thickness

Now let us formulate an expression to the momentum thickness;

The loss of momentum flow of the free stream equals to:

$= \rho_\infty U_\infty \theta \ U_\infty$ (5.5)

The total loss of momentum is given by;

$\int_0^\delta \rho \left(U_\infty - u\right) \ \mathrm{u\ dy}$ (5.6)

By equating the above two equations, we get

$\rho_\infty U_\infty \theta \ U_\infty = \int_0^\delta \rho \left(U_\infty - u\right) \ \mathrm{u\ dy}$

For incompressible flow;$U_\infty \theta * U_\infty = \int_0^\delta \left(U_\infty - u\right) \ \mathrm{u\ dy}$

$\theta = \int_0^\delta \left(U_\infty - u\right) \frac{u}{{U_\infty}^2} \ \mathrm{dy} = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) \frac{u}{U_\infty} \ \mathrm{dy}$

$\theta = \int_0^\delta \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) \ \mathrm{dy}$ (5.7)

$\frac{\theta}{\delta} = \int_0^1 U \left(1 - U\right) \ \mathrm{d}y$

kinetic energy thickness

Finally, an expression for the kinetic energy thickness will be after derivation something like this;

$\frac{\delta^{**}}{\delta} = \int_0^1 U \left(1 - U^2\right) dy$ (5.8)

Momentum equation of hydrodynamics boundary layer over a flat plate

First of all, let us develop the governing equation of the hydrodynamics boundary layer.

The navier-stoke equation in x-direction that derived in section 3;

$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ (3.34)

Assumptions

1. The flow is steady and the fluid is incompressible.
2. The viscosity of the fluid is constant
3. The pressure variation in the direction perpendicular to the flow is negligible.
4. Viscous – shear forces in the y-direction is negligible.
5. Fluid is continuous both in time and space.

After applying the assumptions mentioned before, we get

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$ (6.1)

The equation mentioned above is representing the equation of motion of the momentum equation for hydrodynamics boundary layer.

The next pages, two mathematical solutions will be used to solve the momentum equation for hydrodynamics boundary layer. One of the solutions is approximate which is called Von – Karman and the second is the Blasius Exact solution.

1. **Von-Karman Momentum Integral Equation**
2. **Derivation of the Momentum Integral Equation**

Let us consider control volume of ABCD as in Figure 12 below

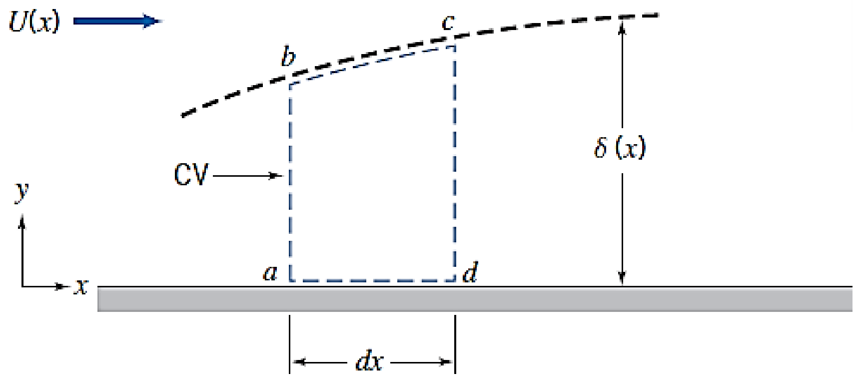

Figure 12 Control volume of Von-Karman Integral Momentum Equation [10]

Mass flow rate entering the c.v. upstream (ab):

$$\dot{m}_1 = \int_0^{\delta} \rho \upsilon \delta \psi \tag{7.1}$$

Mass flow rate leaving the c.v. downstream (dc):

$$\dot{m}_2 = \int_0^{\delta} \rho \upsilon \delta \psi + \frac{d}{\mathrm{dx}} \left[\int_0^{\delta} \rho \upsilon \delta \psi \right] \mathrm{dx} \tag{7.2}$$

The net mass flow rate is

$$\dot{m}_2 - \dot{m}_1 = \int_0^{\delta} \rho \upsilon \delta \psi + \frac{d}{\mathrm{dx}} \left[\int_0^{\delta} \rho \upsilon \delta \psi \right] dx - \int_0^{\delta} \rho \upsilon \delta \psi$$

$$\dot{m}_2 - \dot{m}_1 = \frac{d}{\mathrm{dx}} \left[\int_0^{\delta} \rho \upsilon \delta \psi \right] \mathrm{dx} \tag{7.3}$$

Momentum flux entering ab =

$\int_0^{\delta} \rho u^2 \mathrm{dy}$ (7.4)

Momentum flux entering cd =

$\int_0^\delta \rho u^2 \mathrm{dy} + \frac{d}{\mathrm{dx}}\left[\int_0^\delta \rho u^2 \mathrm{dy}\right]\mathrm{dx}$ (7.5)

Momentum flux entering through bc is given by

$U_\infty * (\dot{m}_2 - \dot{m}_1) = U_\infty \frac{d}{\mathrm{dx}}\left[\int_0^\delta \rho \upsilon \delta \psi\right]\mathrm{dx}$ (7.6)

Now the total drag force must be equal to the rate of change of momentum in flow and out flow;

$$-F_D = \left.\text{Momentum Flux}\right]_{\text{out}} - \left.\text{Momentum Flux}\right]_{\text{in}}$$

$-F_D = \int_0^\delta \rho u^2 \mathrm{dy} + \frac{d}{\mathrm{dx}}\left[\int_0^\delta \rho u^2 \mathrm{dy}\right]\mathrm{dx} - \int_0^\delta \rho u^2 \mathrm{dy} - U_\infty \frac{d}{\mathrm{dx}}\left[\int_0^\delta \rho \upsilon \delta \psi\right]\mathrm{dx}$

$-F_D = \frac{d}{\mathrm{dx}}\left[\int_0^\delta \rho u^2 \mathrm{dy}\right] dx - U_\infty \frac{d}{\mathrm{dx}}\left[\int_0^\delta \rho \upsilon \delta \psi\right]\mathrm{dx}$

$-F_D = \rho \frac{d}{\mathrm{dx}}\left[\int_0^\delta [u^2 - uU_\infty]dy\right]\mathrm{dx}$ (7.7)

Let us multiplying and divided Eq. (7.7) by $\frac{{U_\infty}^2}{{U_\infty}^2}$

$-\tau_w dx = \rho \frac{d}{\mathrm{dx}}\left[\int_0^\delta \frac{{U_\infty}^2}{{U_\infty}^2}[u^2 - uU_\infty]dy\right]\mathrm{dx}$

$\tau_w = \rho {U_\infty}^2 \frac{d}{\mathrm{dx}}\left[\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)\mathrm{dy}\right]$

$\tau_w = \rho {U_\infty}^2 \frac{d}{\mathrm{dx}}\left[\delta \int_0^1 U\left(1 - U\right)dy\right]$ (7.8)

The above expression is the Von-Karman M.I.E. valid for laminar and turbulent shear layer. It has the following form with some manipulation;

Let$I = \frac{\theta}{\delta} = \int_0^1 U\left(1 - U\right)dy$ (7.9)

$\tau_w = \rho {U_\infty}^2 \frac{d}{\mathrm{dx}}\left[\delta \int_0^1 U\left(1 - U\right)dy\right]$ (7.10)

$\tau_w = \rho {U_\infty}^2 \frac{d\theta}{\mathrm{dx}}$ (7.11)

$\tau_w = \rho {U_\infty}^2 I \frac{\mathrm{d}}{\mathrm{dx}}$ (7.12)

Laminar Boundary Layer

There are many Laminar velocity Profiles like the inserted below;

$U = y$ (7.13)

$U = \frac{3}{2}y - \frac{1}{2}y^3$ (7.14)

$U = 2y - y^2$ (7.15)

$U = \sin\frac{\pi}{2}y$ (7.16)

skin friction coefficient

Firstly, an expression of skin friction coefficient will be developed

$C_f = \frac{\tau_w}{\frac{1}{2}\rho {U_\infty}^2} \to \tau_w = \frac{1}{2}\rho {U_\infty}^2 * C_f$ (7.17)

From Von-Karman IME:$\tau_w = \rho {U_\infty}^2 I \frac{\mathrm{d}}{\mathrm{dx}}$ (7.12)

Equate the above Eq. (7.12) with Eq.(7.17)two expression of the shear stress, we find out;

$\frac{1}{2}\rho {U_\infty}^2 * C_f = \rho {U_\infty}^2 I \frac{\mathrm{d}}{\mathrm{dx}}$

$$C_f = 2 * I * \frac{\mathrm{d}}{\mathrm{dx}} = 2\frac{\mathrm{d}}{\mathrm{dx}} \tag{7.13}$$

Rate of growth of L.B.L. over a flat plate

Now we will develop an expression of the rate of growth on flat plate

$C_f = \frac{\tau_w}{\frac{1}{2}\rho {U_\infty}^2}$ (7.17)Also, From Newton's law:$\tau_w = \mu\frac{\partial u}{\partial y}$ (1.1)

$$C_f = \frac{\mu \left.\frac{\mathrm{du}}{\mathrm{dy}}\right]_w}{\frac{1}{2}\rho {U_\infty}^2} \tag{7.18}$$

$C_f = \frac{2\mu}{\rho {U_\infty}^2} \left.\frac{\mathrm{du}}{\mathrm{dy}}\right]_{y=0}$

Since$y = \frac{y}{\delta}$ and $U = \frac{u}{U_\infty}$we will get;

$\left.\frac{\mathrm{du}}{\mathrm{dy}}\right]_{y=0} = \left.\frac{d}{\delta\delta y}\Big(U_\infty U\Big)\right]_{y=0} = \frac{U_\infty}{\delta} \left.\frac{dU}{dy}\right]_{y=0}$

$C_f = \frac{2\mu}{\rho {U_\infty}^2} * \frac{U_\infty}{\delta} \left.\frac{dU}{dy}\right]_{y=0} = \frac{2\mu}{\rho\delta\Upsilon_\infty} \left.\frac{dU}{dy}\right]_{y=0}$

Let us assume dimensionless parameter of velocity profile $D_o = \left.\frac{dU}{dy}\right]_{y=0}$

$$C_f = \frac{2\mu D_o}{\rho\delta\Upsilon_\infty} \tag{7.18}$$

Now let us equalize the Eq. (7.13) with Eq.(7.18) ;$\frac{2\mu D_o}{\rho\delta\Upsilon_\infty} = 2 * I * \frac{\mathrm{d}}{\mathrm{dx}}$

This is 1[st] ODE can be solved simply using the separation of variable method

$\frac{\mathrm{d}}{\mathrm{dx}} = \frac{\mu D_o}{\rho\delta\Upsilon_\infty I}$

$\int \delta\delta\delta = \int \frac{\mu D_o}{\rho U_\infty I}\mathrm{dx} \to \frac{\delta^2}{2} = \frac{\mu D_o}{\rho U_\infty I}x + c$

$at\ x = 0,\ \delta = 0 \to c = 0$

$\frac{\delta^2}{2} = \frac{\mu D_o}{\rho U_\infty I}x$ but$Re = \frac{\rho U_\infty x}{\mu} \to \frac{\rho U_\infty}{\mu} = \frac{\mathrm{Re}}{x}$

$\frac{\delta^2}{2} = \frac{x * D_o}{I * \mathrm{Re}}x \to \delta = \sqrt{\frac{2D_o}{I}}\frac{x}{\sqrt{\mathrm{Re}_x}}$

$$\delta = \sqrt{\frac{2D_o}{I}}\frac{x}{\sqrt{\mathrm{Re}_x}} \tag{7.19}$$

Drag coefficient for flat plate

The drag force is the component of force on a body acting parallel to the direction of motion.

$F_D = w\int_0^L \tau_w \mathrm{dx}$ From one surface

For two upper and lower surface;

$F_D = 2 * \left[w\int_0^L \tau_w \mathrm{dx}\right]$

$F_D = 2wL\tau_w$

The drag coefficient is$C_D = \frac{F_D}{\frac{1}{2}\rho {U_\infty}^2 A} = \frac{2F_D}{\rho {U_\infty}^2 A}$ (7.20)

$C_f = \frac{2\mu D_o}{\rho\delta\Upsilon_\infty}$ (7.18)but $\delta = \sqrt{\frac{2D_o}{I}}\frac{x}{\sqrt{\mathrm{Re}_x}}$

$C_f = \frac{2\mu D_o}{\rho U_\infty * \sqrt{\frac{2D_o}{I}} \frac{x}{\sqrt{\mathrm{Re}_x}}} = \frac{2\mu D_o}{\rho U_\infty x * \sqrt{\frac{2D_o}{I}} \frac{1}{\sqrt{\mathrm{Re}_x}}}$

$$C_f = \sqrt{\frac{2D_o I}{\mathrm{Re}_x}} \tag{7.21}$$

$C_f = \frac{1}{L} \int_0^L C_{f\,x} \mathrm{dx} = C_f = \frac{1}{L} \int_0^L \sqrt{\frac{2D_o I}{\mathrm{Re}_x}} \mathrm{dx} = \frac{1}{L} \sqrt{\frac{2D_o I\mu}{\rho U_\infty}} \int_0^L \frac{\mathrm{dx}}{\sqrt{x}} = \frac{2}{L} \sqrt{\frac{2D_o I\mu}{\rho U_\infty}} \sqrt{L}$

$C_D = 2\sqrt{\frac{2D_o I\mu}{\rho U_\infty L}} = \sqrt{\frac{8D_o I}{\mathrm{Re}_L}} = 2\sqrt{2} \sqrt{\frac{D_o I}{\mathrm{Re}_L}}$

$$C_D = 2 \sqrt{\frac{2D_o I}{\mathrm{Re}_L}} \tag{7.22}$$

$\therefore C_D = 2C_f$

Note:

Blasius is one of the PhD students of Prandtl. The full-detailed of this solution is discussed in Fluid Mechanics books that the reader can read them. However, we will summarized only the obtained formula of

Blasius Exact Solution

Rate of growth of L.B.L. is $\frac{\delta}{x} = \frac{5.0}{\sqrt{\mathrm{Re}_x}}$ (7.23)

Local skin friction coefficient: $C_f = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$ (7.24)

Average skin friction coefficient: $C_f = C_D = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$ (7.25)

As an example to explain the laminar boundary layer, let us assume we have the simplest laminar velocity profile which is $U = y$

We shall use the rate of growth formula inserted below;

$\frac{\delta}{x} = \sqrt{\frac{2D_o}{\mathrm{I}\ \mathrm{Re}_x}}$

$D_o = \frac{dU}{dy}\Big]_{y=0} = 1$

$I = \int_0^1 U \left(1 - U\right) dy = \int_0^1 y \left(1 - y\right) dy = \frac{1}{6}$

Then, $\frac{\delta}{x} = \frac{3.464101615}{\sqrt{\mathrm{Re}_x}}$

Let us obtained the local skin friction coefficient using the formula inserted below;

$C_{f\,x} = 2 * I * \frac{\mathrm{d}}{\mathrm{dx}} = 2 * \frac{1}{6} \frac{d}{\mathrm{dx}} \left\{ \frac{3.464101615}{\sqrt{\mathrm{Re}_x}} x \right\} = \frac{1}{3} \frac{d}{\mathrm{dx}} \left\{ 3.464101615 \frac{\nu^{0.5}}{U_\infty{}^{0.5} x^{0.5}} x \right\}$

$C_{f\,x} = \frac{1}{3} * 3.646 \left(\frac{\nu}{U_\infty}\right)^{0.5} \frac{d}{\mathrm{dx}} x^{0.5} = \frac{1}{3} * 3.646 * 0.5 \left(\frac{\nu}{U_\infty}\right)^{0.5} 1/x^{0.5} = \frac{\sqrt{3}}{3} \left(\frac{\nu}{U_\infty x}\right)^{0.5}$

$C_{f\,x} = \frac{0.5773502692}{\sqrt{\mathrm{Re}_x}}$, Commonly written as $C_{f\,x} = \frac{0.577}{\sqrt{\mathrm{Re}_x}}$

The drag coefficient is more convenient in the aerodynamics researches, in this way, we shall find out its expression for this profile;

$C_f = \frac{1}{L} \int_0^L C_{f\,x} \mathrm{dx} = \frac{1}{L} \int_0^L \frac{0.577}{\sqrt{\mathrm{Re}_x}} \mathrm{dx} = \frac{1}{L} \int_0^L 0.577 \frac{\nu^{0.5}}{U_\infty{}^{0.5} x^{0.5}} \mathrm{dx}$

$C_f = \frac{0.577}{L} \left(\frac{\nu}{U_\infty}\right)^{0.5} \int_0^L x^{-0.5} \mathrm{dx} = \frac{0.577}{L} \left(\frac{\nu}{U_\infty}\right)^{0.5} \left.\frac{x^{0.5}}{0.5}\right|_0^L = \frac{0.577}{0.5} \left(\frac{\nu}{U_\infty}\right)^{0.5} \frac{L^{0.5}}{L}$

$$C_f = 1.154700538 * \left(\frac{\nu}{U_\infty L}\right)^{0.5} = \frac{1.1547}{\sqrt{\mathrm{Re}_L}}$$

The same procedure can be used for other laminar velocity profile. We tableted the famous velocity profiles and their characteristics below in Table 2;

Table 2 Velocity Profile of Laminar Boundary Layer Characteristics

Velocity profile	**Rate of growth δ/x**	**Drag Coefficient $\mathbf{C_f}$**
$U = y$	$\frac{3.464101615}{\sqrt{\mathrm{Re}_x}}$	$\frac{1.1547}{\sqrt{\mathrm{Re}_L}}$
$U = \frac{3}{2}y - \frac{1}{2}y^3$	$\frac{4.64}{\sqrt{\mathrm{Re}_x}}$	$\frac{1.292}{\sqrt{\mathrm{Re}_L}}$
$U = 2y - y^2$	$\frac{5.48}{\sqrt{\mathrm{Re}_x}}$	$\frac{1.46}{\sqrt{\mathrm{Re}_L}}$
$U = \sin\frac{\pi}{2}y$	$\frac{4.795}{\sqrt{\mathrm{Re}_x}}$	$\frac{1.31}{\sqrt{\mathrm{Re}_L}}$
Blasius Exact Solution	$\frac{5}{\sqrt{\mathrm{Re}_x}}$	$\frac{1.328}{\sqrt{\mathrm{Re}_L}}$

Turbulent boundary layer

Unlike the L.B.L. there is only one well-known turbulent velocity profile which is known as the seventh root law profile that suggested by the Prandtl:

$$U = y^{\frac{1}{7}} \qquad (7.26)$$

Local stream-function coefficient

Let us formulate an expression of the local stream-function coefficient in turbulent boundary layer.

$$\tau_w = 0.0233\rho U^{\frac{7}{4}}\left(\frac{\nu}{y}\right)^{0.25} \qquad (7.27)$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho {U_\infty}^2} = \frac{0.0233\rho U^{\frac{7}{4}}\left(\frac{\nu}{y}\right)^{0.25}}{\frac{1}{2}\rho {U_\infty}^2} = \frac{0.0466\ \rho U^{\frac{7}{4}}\left(\frac{\nu}{y}\right)^{0.25}}{\rho {U_\infty}^2}$$

$at\ y = \delta\ \rightarrow U = U_\infty$

$$C_f = \frac{0.0466\ \rho {U_\infty}^{\frac{7}{4}}\left(\frac{\nu}{y}\right)^{0.25}}{\rho {U_\infty}^2} = \frac{0.0466\nu^{0.25}}{{U_\infty}^{0.25}\delta^{0.25}} = \frac{0.0466}{\mathrm{Re}^{0.25}{}_\delta}$$

$$C_f = \frac{0.0466}{{\mathrm{Re}_\delta}^{0.25}} \qquad (7.28)$$

Rate of growth of turbulent boundary layer

The derivation is starting by writing the M.I.E. ;

$$\tau_w = \rho {U_\infty}^2 \frac{d\theta}{\mathrm{dx}}$$

$$\tau_w = \rho {U_\infty}^2 I \frac{\mathrm{d}}{\mathrm{dx}}$$

$$I = \int_0^1 U\left(1 - U\right)dy = \int_0^1 y^{\frac{1}{7}}\ \left(1 - y^{\frac{1}{7}}\right)dy = \frac{7}{72}$$

$$\tau_w = \frac{7}{72}\rho {U_\infty}^2 \frac{\mathrm{d}}{\mathrm{dx}} \rightarrow \frac{\tau_w}{\rho {U_\infty}^2} = \frac{7}{72}\frac{\mathrm{d}}{\mathrm{dx}} \qquad (7.29)$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho {U_\infty}^2} \rightarrow \frac{1}{2}C_f = \frac{\tau_w}{\rho {U_\infty}^2} \qquad (7.17)$$

Let us equalize Eq. (7.17) with Eq. (7.29), we get $\frac{7}{72}\frac{\mathrm{d}}{\mathrm{dx}} = \frac{1}{2}C_f$ (7.30)

But we have already obtained $C_f = \frac{0.0466}{{\mathrm{Re}_\delta}^{0.25}}$ (7.28)

Substitute Eq. (7.28) in Eq. (7.30)

$\frac{1}{2}\left\{\frac{0.0466}{{\mathrm{Re}_\delta}^{0.25}}\right\} = \frac{7}{72}\frac{\mathrm{d}}{\mathrm{dx}} \to \frac{1}{2}\frac{0.0466\nu^{0.25}}{{U_\infty}^{0.25}\delta^{0.25}} = \frac{7}{72}\frac{\mathrm{d}}{\mathrm{dx}}$ This is 1st ODE will be solved simply using separation of variable method

$$\int_0^x 0.23965\left(\frac{\nu}{U_\infty}\right)^{0.25} \mathrm{dx} = \int_0^{0.25} \delta\delta\delta \to \frac{4}{5}\delta^{5/4} = 0.23965\left(\frac{\nu}{U_\infty}\right)^{1/4} \mathrm{x}$$

$$\delta = 0.3812325351\left(\frac{\nu}{U_\infty}\right)^{1/5} x^{4/5} = 0.3812325351\left(\frac{\nu}{U_\infty}\right)^{0.2}\frac{x}{x^{0.2}}$$

$$\delta = \frac{0.3812325351x}{{\mathrm{Re}_x}^{0.2}}$$

$$\frac{\delta_T}{x} = \frac{0.38123}{{\mathrm{Re}_x}^{0.2}} \tag{7.31}$$

1. **Characteristics of turbulent boundary layer**
2. **Displacement thickness**

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - U\right) dy$$

$$U = y^{\frac{1}{7}}$$

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - y^{\frac{1}{7}}\right) dy = \frac{1}{8}$$

$$\delta^* = \frac{1}{8} * \delta = \frac{1}{8} * \frac{0.38123x}{{\mathrm{Re}_x}^{0.2}} = \frac{0.04765}{{\mathrm{Re}_x}^{0.2}}\ \mathrm{x}$$

$$\delta^* = \frac{0.04765}{{\mathrm{Re}_x}^{0.2}}\ \mathrm{x} \tag{7.32}$$

Momentum thickness

$$\frac{\theta}{\delta} = \int_0^1 U\left(1 - U\right) dy = \int_0^1 y^{\frac{1}{7}}\left(1 - y^{\frac{1}{7}}\right) dy = \frac{7}{72}$$

Similarly, we get $\theta = \frac{0.03706}{{\mathrm{Re}_x}^{0.2}}\ \mathrm{x}$ (7.33)

Energy thickness

By the same approach, we shall get $\delta^{**} = \frac{0.0667}{{\mathrm{Re}_x}^{0.2}}\ \mathrm{x}$ (7.34)

Skin friction coefficient for turbulent boundary layer

$$C_f = \frac{0.0466}{{\mathrm{Re}_\delta}^{0.25}} \tag{7.28}$$

$$C_f = \frac{0.0466\nu^{0.25}}{{U_\infty}^{0.25}\delta^{0.25}} = 0.0466 * \left(\frac{\nu}{U_\infty}\right)^{0.25} * \left(\frac{1}{\delta}\right)^{0.25}$$

$$\frac{\delta}{x} = \frac{0.38123}{{\mathrm{Re}_x}^{0.2}} \to \quad \delta = \frac{0.38123}{{\mathrm{Re}_x}^{0.2}}\mathrm{x} \tag{7.31}$$

Substitute Eq. (7.31) into Eq. (7.28), results;

$$C_f = 0.0466 * \left(\frac{\nu}{U_\infty}\right)^{0.25} * \left(\frac{1}{\frac{0.38123}{{\mathrm{Re}_x}^{0.2}}x}\right)^{0.25} = 0.0466 * \left(\frac{\nu}{U_\infty}\right)^{0.25} * \frac{{\mathrm{Re}_x}^{0.2^{0.25}}}{0.38123^{0.25}\ \mathrm{x}^{0.25}}$$

$$C_f = 0.05930470493 * \left(\frac{\nu}{U_\infty}\right)^{0.25}\frac{{\mathrm{Re}_x}^{0.05}}{x^{0.25}} = 0.05930470493 * \left(\frac{\nu}{U_\infty x}\right)^{0.25}{\mathrm{Re}_x}^{0.05}$$

$$C_{f\,x} = \frac{0.05930470493}{{\mathrm{Re}_x}^{0.2}} \tag{7.35}$$

Finally, the wall shear stress will be;

$$\tau_w = \frac{1}{2}\rho {U_\infty}^2 C_{f\,x} = \frac{1}{2}\rho {U_\infty}^2\left\{\frac{0.05930}{{\mathrm{Re}_x}^{0.2}}\right\} = 0.02965 * \frac{\rho {U_\infty}^2}{{\mathrm{Re}_x}^{0.2}} \tag{7.36}$$

Mixed (Transition) Boundary Layer Region

The schematic diagram of the external flow over a flat plate is inserted below;

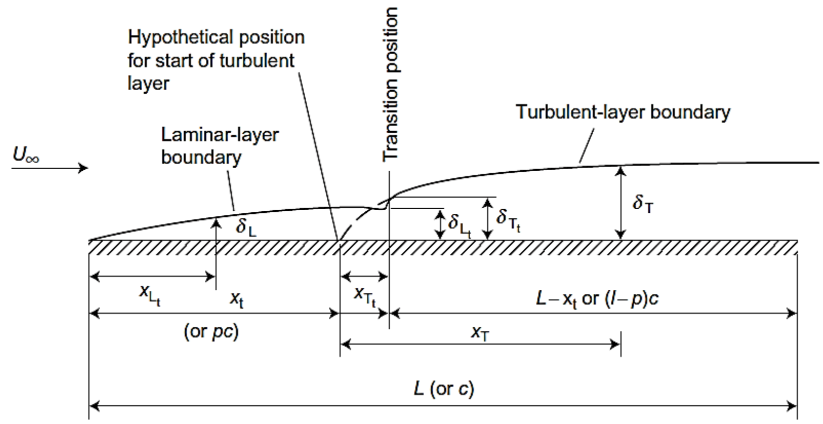

Figure 13 External flow illustrating the transition zone [11]

Firstly, let us develop an expression of turbulent boundary layer thickness at the transition region in terms of Reynolds number at the laminar boundary layer region as explained below;

However, for laminar zone, sine profile will be selected as a special case study;$U = \sin \frac{\pi}{2} y$

$$\delta_{L,t} = \frac{4.795\ x_{L,t}}{\sqrt{\mathrm{Re}_{L,t}}} = 4.79 \left(\frac{\nu}{U_\infty\ x_{L,t}}\right)^{0.5} x_{L,t} = 4.79 \left(\frac{\nu}{U_\infty}\right)^{0.5} {x_{L,t}}^{0.5} \quad (7.37)$$

Multiplied and divided the above equation by$\left(\frac{U_\infty}{\nu}\right)^{1/2}$

$$\delta_{L,t} = \frac{4.795\ x_{L,t}}{\sqrt{\mathrm{Re}_{L,t}}} = 4.79 \left(\frac{\nu}{U_\infty}\right)^{0.5} {x_{L,t}}^{0.5} * \left(\frac{U_\infty}{}\right)^{\frac{1}{2}} \left(\frac{\nu}{U_\infty}\right)^{\frac{1}{2}}$$

$$\delta_{L,t} = 4.79 \left(\frac{\nu}{U_\infty}\right)^{1/2} {\mathrm{Re}_{L,t}}^{0.5} \quad (7.38)$$

Thus,$\delta_{T,t} = 1.4\delta_{L,t}$ (7.39)

$$\delta_{T,t} = 1.4 * 4.79 \left(\frac{\nu}{U_\infty}\right)^{\frac{1}{2}} {\mathrm{Re}_{L,t}}^{0.5} = 6.706 \left(\frac{\nu}{U_\infty}\right)^{\frac{1}{2}} {\mathrm{Re}_{L,t}}^{0.5}$$

$$\delta_{T,t} = 6.706 \left(\frac{\nu}{U_\infty}\right)^{\frac{1}{2}} {\mathrm{Re}_{L,t}}^{0.5} \quad (7.40)$$

Secondly, let us find out a general expression of turbulent layer in the transition region;

$$\delta_{T,t} = \frac{0.38123\ x_{T,t}}{{\mathrm{Re}_{T,t}}^{0.2}} = \frac{0.38123\ x_{T,t}}{\left(\frac{U_\infty\ \xi_{T,t}}{\nu}\right)^{0.2}} = 0.38123 \left(\frac{\nu}{U_\infty}\right)^{0.2} {x_{T,t}}^{0.8} \quad (7.41)$$

By equalizing Eq. (7.40) with Eq. (7.41);

$$6.706 \left(\frac{\nu}{U_\infty}\right)^{\frac{1}{2}} {\mathrm{Re}_{L,t}}^{0.5} = 0.38123 \left(\frac{\nu}{U_\infty}\right)^{0.2} {x_{T,t}}^{0.8}$$

$${x_{T,t}}^{0.8} = \frac{6.706}{0.38123} \left(\frac{\nu}{U_\infty}\right)^{0.8} {\mathrm{Re}_{L,t}}^{0.5}$$

$$x_{T,t} = 36.023469\ \frac{\nu}{U_\infty}\ {\mathrm{Re}_{L,t}}^{5/8} \quad (7.42)$$

Finally, as aerodynamics researchers we are more interest in obtaining a formula for drag coefficient at the transition zone;

The effective length of turbulent layer is given by;$L_T = L - X_t + X_{T,t}$

$D = \int_0^{L-X_t+X_{T,t}} \tau_w \text{ dx} = \frac{1}{2}\rho {U_\infty}^2 \int_0^{L-X_t+X_{T,t}} C_f \text{dx}$

The local skin friction coefficient in the turbulent region can be obtained from the equation inserted below;

$C_f = \frac{0.0593}{{\mathrm{Re}_x}^{0.2}}$

$D = \frac{1}{2}\rho {U_\infty}^2 \int_0^{L-X_t+X_{T,t}} \frac{0.0593}{{\mathrm{Re}_x}^{0.2}} \text{dx}$

$D = \frac{0.0593}{2}\rho {U_\infty}^2 \int_0^{L-X_t+X_{T,t}} \frac{\nu^{0.2}}{(U_\infty \ x)^{0.2}} \text{dx}$

$D = \frac{0.0593}{2}\rho {U_\infty}^2 \int_0^{L-X_t+X_{T,t}} \left(\frac{\nu}{U_\infty}\right)^{0.2} \ x^{-0.2} \text{dx}$

$D = \frac{0.0593}{2}\rho {U_\infty}^2 \left(\frac{\nu}{U_\infty}\right)^{0.2} \int_0^{L-X_t+X_{T,t}} \ x^{-0.2} \text{dx}$

$D = \frac{0.0593}{2}\rho {U_\infty}^2 \left(\frac{\nu}{U_\infty}\right)^{0.2} \frac{(L-X_t+X_{T,t})^{0.8}}{0.8}$

$$D = \frac{0.0593}{2*0.8}\rho {U_\infty}^2 \left(\frac{\nu}{U_\infty}\right)^{0.2} (L - X_t + X_{T,t})^{0.8} \quad (7.43)$$

Let us first simplify $(L - X_t + X_{T,t})^{0.8}$ by multiplying and divided it by$\left(\frac{U_\infty}{\nu}\right)$

$(L - X_t + X_{T,t})^{0.8} = \left(\frac{\nu}{U_\infty} * \frac{U_\infty}{\nu} (L - X_t + X_{T,t})\right)^{0.8} = \left(\frac{\nu}{U_\infty} * (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})\right)^{0.8}$

$\therefore D = \frac{0.0593}{2*0.8}\rho {U_\infty}^2 \left(\frac{\nu}{U_\infty}\right)^{0.2} \left(\frac{\nu}{U_\infty} * (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})\right)^{0.8}$

$\therefore D = \frac{0.0593}{2*0.8}\rho {U_\infty}^2 \left(\frac{\nu}{U_\infty}\right)^{0.2} \left(\frac{\nu}{U_\infty}\right)^{0.8} (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})^{0.8}$

$\therefore D = \frac{0.0593}{2*0.8}\rho {U_\infty}^2 \frac{\nu}{U_\infty} (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})^{0.8}$

$$\therefore D = 0.0370625 \ \rho U_\infty \nu (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})^{0.8} \quad (7.45)$$

The drag coefficient is nothing but the drag divided by$\frac{1}{2}\rho {U_\infty}^2 S$

$C_d = \frac{D}{\frac{1}{2}\rho {U_\infty}^2 S} = \frac{0.0370625 \ \rho U_\infty \nu (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})^{0.8}}{\frac{1}{2}\rho {U_\infty}^2 L*1}$

$C_d = \frac{2*0.0370625 \ \nu (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})^{0.8}}{U_\infty L}$

$C_d = 0.074125 \left(\frac{}{U_\infty L}\right) (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})^{0.8}$

$C_d = \frac{0.074125}{\mathrm{Re}_L} (\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t})^{0.8}$

Since $\mathrm{Re}_{T,t} = \frac{U_\infty \ X_{T,t}}{\nu}$

The equivalent length of turbulent layer in the transition zone($X_{T,t}$) is given for sine laminar profile as derived previously,

$x_{T,t} = 36.023469 \ \frac{\nu}{U_\infty} \ {\mathrm{Re}_{L,t}}^{5/8}$

$\mathrm{Re}_{T,t} = \frac{U_\infty}{\nu} X_{T,t} = \frac{U_\infty}{\nu} 36.023469 \ \frac{\nu}{U_\infty} \ {\mathrm{Re}_{L,t}}^{5/8}$

$\mathrm{Re}_{T,t} = 36.023469 \ \ {\mathrm{Re}_{L,t}}^{5/8}$

In this way, we finally get

$C_d = \frac{0.074125}{\mathrm{Re}_L}\left(\mathrm{Re}_L - \mathrm{Re}_t + \mathrm{Re}_{T,t}\right)^{0.8}$

$$C_d = \frac{0.074125}{\mathrm{Re}_L}\left(\mathrm{Re}_L - \mathrm{Re}_t + 36.023469\ \ {\mathrm{Re}_{L,t}}^{5/8}\right)^{0.8} \tag{7.46}$$

Chapter Summary and study guide

The present chapter demonstrates the mathematical analysis of the hydrodynamics boundary layer over a flat plate. The main important points can be summarized in the following points:

- The boundary layer over a flat plate are three regions :laminar, transition and turbulent region. It is worthy to mention that the transition region analysis in the text book is limited and we hope the explanation of the transition region in the present chapter could help to the engineering students in better understanding of this region.
- The governing equations of fluid mechanics :mass, energy and momentum of fluid had been derived in – full details.
- The turbulence is also illustrated in this chapter with mention of important applications of the turbulent flow.
- The governing equations of hydrodynamics boundary layer had been investigated also and then Von – Karman solution for this equation is derived.
- We starting from the M.I.E. to derive an expression of the boundary layer thickness, drag, skin friction coefficient, drag coefficient, shear wall stress in terms of Reynolds number for laminar, transition and turbulent regions.

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