A general curvilinear magnetic field-line-following coordinate system for ionosphere-plasmasphere modeling

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Key Points:

- We propose a general curvilinear field-line-following coordinate system which reduces to a usual dipole coordinate system for a dipole field
- Highly accurate results are obtained using the high-order ODE solver to solve the general magnetic field line equations
- Numerical results show that this general curvilinear coordinate system is also an Euler potential or Clebsch type coordinate system

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Abstract

We propose a simple way to define a field-line-following, general curvilinear coordinate system for a general magnetic field. This definition of field-line-following coordinate system reduces to a usual definition of dipole coordinate system when the magnetic field is approximated by an axisymmetric dipole. In this way, it can facilitate the numerical implementation by enabling validation of various metric terms computed numerically against those defined analytically in the case of dipole field. Steps involved in grid generation are also sketched. Highly accurate results are obtained using the high-order ordinary differential equation (ODE) solver to solve the general magnetic field line equations. The accuracy and consistency of the numerical implementation are validated against analytical results in the case of a dipole field. Numerical results show that this field-line-following coordinate system for the general magnetic field, like the coordinates for the dipole field, is also an Euler potential or Clebsch type coordinate system.

1 Introduction

A coordinate system provides a way of organizing data. What kind of coordinate systems to use depends on the kind of problems to be solved. Because the fundamental role played by the geomagnetic main fields in plasma motion in ionosphere and plasmasphere , it is often advantageous to organize ionosphere-plasmasphere data or define model variables along the magnetic field lines.

Several magnetic coordinate systems have been proposed for ionospheric studies in the past, see Laundal & Richmond (2017) for a comprehensive review. The apex-based coordinates, such as the apex coordinates (VanZandt et al. (1972)), the modified apex coordinates and quasi-dipole coordinates (Richmond (1995)), were discussed in Emmert et al. (2010), along with the computational aspects of apex-type coordinates. Use of the Euler potentials as coordinate variables is also appealing because of the special properties of the Euler potentials (e.g., Stern (1967, 1970)).

The Earth's main magnetic field can be best described by the International Geomagnetic Reference Field (IGRF) (Thébault et al. (2015)). Approximated magnetic field such as the eccentric dipole, a tilted dipole with an offset center, is sometimes used in the ionospheric modeling (e.g., Bailey et al. (1993); Huba et al. (2000)).

It would be a helpful aid for ionosphere-plasmasphere model development to use the general curvilinear coordinate system that is *consistent* with the conventional mathematical notations. In this paper, we propose and derive a magnetic field-line-following coordinate system that is consistent with the idea and notation of the rigorous mathematics of general curvilinear coordinates. This definition also reduces to a usual definition of the dipole coordinate system in the case of a dipole field.

This paper is organized as follows. In the next section, we present the definition of the general field-line-following coordinate system. Computation of the basis vectors are described in section 3. Evaluation of numerical implementation and computational results are given in section 4. And the final section is a summary. Some mathematical details related to algorithm development are presented in the appendices, where grid generation procedures are also briefly described.

2 Definition of coordinate systems for the magnetic field

2.1 A coordinate system for the dipole magnetic field

The simplest configuration of a magnetic field is a dipole. To aid our discussion, we start with a definition of a dipole coordinate system. Using the notation of Kageyama et al. (2006), the dipole coordinates (μ, χ, ϕ) for an axial-centered dipole field in terms of the spherical polar coordinates (r, θ, ϕ) are defined as

$$\mu = -\frac{\cos\theta}{r^2}, \quad \chi = \frac{\sin^2\theta}{r}, \quad \phi = \phi, \tag{1}$$

where r is the radial distance from Earth's center, normalized by the geomagnetic conventional Earth's mean reference spherical radius $a = 6371.2$ km, θ the geocentric colatitude, and ϕ the east longitude. The coordinate μ is a magnetic scalar potential function for the dipole field, and the magnetic flux density is given by $\mathbf{B} = -m\nabla\mu$, with m as the dipole moment. The dipole coordinates (μ, χ, ϕ) are *orthogonal*.

It can be shown that

$$\nabla\chi \times \nabla\phi = \nabla\mu.$$

Hence $\mathbf{B} = -m\nabla\mu = -m\nabla\chi \times \nabla\phi$. Thus, the coordinates χ, ϕ are the *Euler potentials* (e.g., Stern, 1970). They are also called the Clebsch-type coordinates (D'haeseleer et al., 1991, Chapter 5).

2.2 A general curvilinear magnetic field-line-following coordinate system

We would like to define a general magnetic field-line-following coordinate system in such a way that, when the magnetic field becomes a dipole, the definition *seamlessly and naturally* becomes the definition of coordinates for a dipole field, the (μ, χ, ϕ) coordinates. Thus, we propose to define a magnetic coordinate system (μ_m, χ_m, ϕ_m) as follows:

$$\mu_m = \hat{\Phi}, \quad \chi_m = \frac{\sin^2\theta_m}{r}, \quad \phi_m = \phi_A, \tag{2}$$

where $\hat{\Phi}$ is a magnetic scalar potential (more details later), θ_m is the magnetic colatitude defined by

$$\frac{\sin^2\theta_m}{r} = \frac{1}{r_A},$$

with r_A the radial distance to the apex (a *constant* for each field line), and ϕ_A is the (geographic) longitude at the apex. Both r_A and ϕ_A are *uniquely* defined for each field line, *as long as each field line has a unique, well-defined apex.* Thus, they can be used to *label* each field line; hence (μ_m, χ_m, ϕ_m) as defined in Eq. (2) can be used as the coordinate variables.

The *shifted* and *normalized* magnetic scalar potential $\hat{\Phi}$ is defined as follows:

$$\hat{\Phi} \equiv \frac{\Phi - \Phi_A}{g_m},$$

where Φ is the magnetic scalar potential, Φ_A is the magnetic potential at the apex, and g_m is the dipole moment used here as the normalization factor. Thus the coordinate $\mu_m \equiv \hat{\Phi}$ is defined in such way that the origin of the coordinate is at the apex. The magnetic flux density is then given by $\mathbf{B} \equiv -\nabla\Phi = -g_m\nabla\hat{\Phi}$. For the International Geomagnetic Reference Field (IGRF), Φ and g_m are defined in Eqs. (A1) and (A3), respectively.

3 Basis vectors and metric terms

Two sets of basis vectors can be defined for a general curvilinear coordinate system. The contravariant-basis vectors are defined as the gradient of the coordinate variables, while the covariant-basis vectors are tangent to the coordinate curves. The two sets of basis vectors are *reciprocal sets of vectors*: one can derive one set of the basis vectors once the other set is known or vice versa (D'haeseleer et al., 1991, Chapter 2).

3.1 Basis vectors and metric terms for a dipole field

For a dipole field, the metric terms can be derived analytically. The contravariant-basis vectors are the gradient of the coordinate variables (μ, χ, ϕ), which can be written in terms of the spherical coordinates as

$$\mathbf{e}^\mu \equiv \nabla\mu = \frac{2\cos\theta}{r^3}\hat{\mathbf{r}} + \frac{\sin\theta}{r^3}\hat{\theta}, \tag{3a}$$

$$\mathbf{e}^\chi \equiv \nabla\chi = -\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}} + \frac{2\sin\theta\cos\theta}{r^2}\hat{\theta}, \tag{3b}$$

$$\mathbf{e}^\phi \equiv \nabla\phi = \frac{1}{r\sin\theta}\hat{\phi}, \tag{3c}$$

where $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi})$ are unit vectors of the spherical polar coordinates (r, θ, ϕ). The covariant-basis vectors, as the reciprocal of the contravariant-basis vectors, can be computed from the contravariant-basis vectors as follows:

$$\mathbf{e}_\mu = \frac{\mathbf{e}^\chi \times \mathbf{e}^\phi}{\mathbf{e}^\mu \cdot (\mathbf{e}^\chi \times \mathbf{e}^\phi)} = \frac{2r^3\cos\theta}{\Theta^2}\hat{\mathbf{r}} + \frac{r^3\sin\theta}{\Theta^2}\hat{\theta}, \tag{4a}$$

$$\mathbf{e}_\chi = \frac{\mathbf{e}^\phi \times \mathbf{e}^\mu}{\mathbf{e}^\chi \cdot (\mathbf{e}^\phi \times \mathbf{e}^\mu)} = -\frac{r^2}{\Theta^2}\hat{\mathbf{r}} + \frac{2r^2}{\Theta^2\tan\theta}\hat{\theta}, \tag{4b}$$

$$\mathbf{e}_\phi = \frac{\mathbf{e}^\mu \times \mathbf{e}^\chi}{\mathbf{e}^\phi \cdot (\mathbf{e}^\mu \times \mathbf{e}^\chi)} = r\sin\theta\hat{\phi}, \tag{4c}$$

where Θ is defined as

$$\Theta = \sqrt{1 + 3\cos^2\theta}.$$

The *scale factors* can be computed as

$$h_\mu = |\mathbf{e}_\mu| = 1/|\nabla\mu| = r^3/\Theta, \tag{5a}$$

$$h_\chi = |\mathbf{e}_\chi| = 1/|\nabla\chi| = r^2/(\Theta\sin\theta), \tag{5b}$$

$$h_\phi = |\mathbf{e}_\phi| = 1/|\nabla\phi| = r\sin\theta. \tag{5c}$$

And, in terms of the scale factors, the differential arc length ds is given by

$$ds^2 = ds_\mu^2 + ds_\chi^2 + ds_\phi^2 = (h_\mu d\mu)^2 + (h_\chi d\chi)^2 + (h_\phi d\phi)^2. \tag{6}$$

For a dipole magnetic field, an analytical expression for the arc length can be obtained, see Eq. (B2). We will compare the arc length computed using Eqs. (B2), (6), and (7) of the general coordinates in the case of a dipole field; see section 4.1, especially Table 2.

3.2 Basis vectors and metric terms for a general magnetic field

For the general magnetic field, the basis vectors and metric terms can only be computed numerically. We first compute contravariant-basis vectors as the gradient of the coordinate variables (e.g., D'haeseleer et al., 1991):

$$\mathbf{e}^i \equiv \nabla u^i.$$

Then, the covariant-basis vectors can be computed from the contravariant-basis vectors as follows:

$$\mathbf{e}_i = \frac{\mathbf{e}^j \times \mathbf{e}^k}{\mathbf{e}^i \cdot (\mathbf{e}^j \times \mathbf{e}^k)},$$

where i, j and k are chosen such that (i, j, k) forms a cyclic permutation of $(1, 2, 3)$. We will use the correspondence notation $(1, 2, 3) \Leftrightarrow (\mu_m, \chi_m, \phi_m)$.

Two important metric coefficients g_{ij} and g^{ij} are defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \quad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j.$$

The off-diagonal metric coefficients are all zero for an orthogonal but not necessarily orthonormal coordinate system. The scale factors are defined as $h_i = |\mathbf{e}_i|$. Thus

$$g_{ii} = h_i^2 \quad \text{or} \quad h_i = \sqrt{g_{ii}},$$

and so h_i's are also called metric coefficients; no summation rule is implied here. Although the h_i's are usually used for orthogonal coordinate systems, the above definition is valid for any coordinate system (D'haeseleer et al., 1991). The arc length can then be computed the same as in the dipole case:

$$ds^2 = ds_{\mu_m}^2 + ds_{\chi_m}^2 + ds_{\phi_m}^2 = (h_1 d\mu_m)^2 + (h_2 d\chi_m)^2 + (h_3 d\phi_m)^2, \tag{7}$$

where $h_1 = h_{\mu_m}, h_2 = h_{\chi_m}, h_3 = h_{\phi_m}$, using the correspondence notation $(1, 2, 3) \Leftrightarrow (\mu_m, \chi_m, \phi_m)$.

4 Grid generation and computational results

Grid generation is an important step in ionosphere-plasmasphere model development. It is usually non-trivial, especially for the general magnetic field-line-following coordinates. The procedures of grid generation are briefly described in Appendix C. In this section we evaluate the accuracy of numerical implementation of the algorithms for various coordinate variables and metric terms.

4.1 Evaluation in the case of a dipole field

Because of the way we define the general coordinate system, we find that the validation of the implementation and the assessment of numerical algorithms can be conveniently performed in the case of a dipole field. This is done as follows:

(1) For dipole coordinates, grid generation is done as described in C1. The basis vectors and metric terms are computed using analytical expressions given in section 3.1.
(2) For general coordinates, grid generation is done as described in C2. Note in choosing/specializing the dipole field from the IGRF model, instead of using all the terms as in Eq. (A1), only one term, the g_1^0 term, is used as in Eq. (A2). The basis vectors and metric terms are then computed as described in section 3.2.

We choose the field line crossing the earth's surface at colatitude $\theta = 45°$ at longitude $\phi = 0°$. We compute the relative errors of numerically computed values of spherical coordinate variables (radial distance r and latitude $\varphi = \pi/2 - \theta$), magnetic flux density B, and scale factors h_1, h_2 and h_3, based on the procedure (2) above, relative to their corresponding values based on procedure (1) in the case of axial symmetric dipole field. The relative error $\mathcal{E}_x$ of a variable x is defined as

$$\mathcal{E}_x = (x - x_0)/x_0,$$

where x_0 is expected value of the variable x. Figure 1 shows the results. They are plotted as the function of the arc length/distance along the field line.

First, notice that these errors are all very small; see also Table 1, which lists the minimum and maximum of these relative errors.

Another noticeable feature is that these errors are symmetric about the apex point. This symmetry in the case of dipole field is a good indicator of the consistency and accuracy of the numerical implementation of the algorithms.

We also calculate the relative errors of various ways of computing the arc length for the field line. We call the arc length calculated using the scale factors from Eq. (6) and Eq. (7) the *discretized arc length*, denoted by s_{dc} and s_{gc}, respectively; while the arc

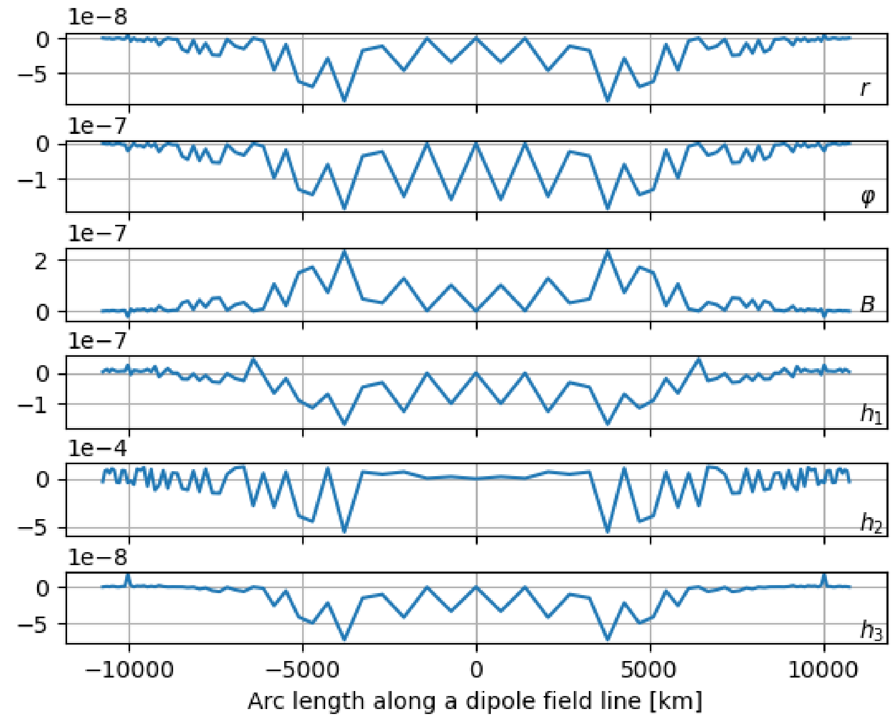

Figure 1. The relative errors of numerically computed spherical coordinate variables (r, φ), magnetic flux density (B), and scale factors (h_1, h_2, h_3) with respect to those computed using analytical expressions in the case of a dipole field line.

length calculated using the analytical expression Eq. (B2) the *continuous arc length*, denoted by ℓ. Table 2 shows the computed arc lengths [km] of a dipole field line, and their relative errors: w.r.t. s_{dc} ($\mathcal{E}_{gc/dc}$) or w.r.t. ℓ ($\mathcal{E}_{gc/\ell}$ and $\mathcal{E}_{dc/\ell}$). Again these errors are very small, indicating the high accuracy of the numerical scheme and robustness of the numerical implementation.

4.2 Into the general magnetic field

For the general magnetic field, specifically the IGRF used in this study, we first check the orthogonality of the coordinates. For this we compute the angles between the contravariant basis vectors. The angle between the basis vectors $\mathbf{e}^i$ and $\mathbf{e}^j$, denoted by α_{ij}, can be computed from their dot product

$$\mathbf{e}^i \cdot \mathbf{e}^j = |\mathbf{e}^i||\mathbf{e}^j| \cos(\alpha_{ij}).$$

Figure 2 shows the departures from 90° of the angles between the three contravariant basis vectors

$$\beta_{ij} \equiv \alpha_{ij} - 90°,$$

with their minima and maxima listed in Table 3, showing that both β_{12} and β_{13} are very small. Thus, the two basis vectors $\mathbf{e}^2$ and $\mathbf{e}^3$, though not orthogonal to each other themselves (β_{23} not small), they are both *perpendicular* to $\mathbf{e}^1$, i.e., the magnetic field. Here we use the correspondence notation $(1, 2, 3) \Leftrightarrow (\mu_m, \chi_m, \phi_m)$ again.

Table 1. The minima and maxima of the relative errors $\mathcal{E}$ of spherical coordinate variables (r, φ), magnetic flux density (B), and metric coefficients (h_1, h_2, h_3) for a dipole field line.

$\mathcal{E}$	Min	Max
r	-8.9068×10^{-8}	3.6699×10^{-9}
φ	-1.8329×10^{-7}	4.9260×10^{-12}
B	-2.0742×10^{-8}	2.3209×10^{-7}
h_1	-1.6877×10^{-7}	4.5553×10^{-8}
h_2	-5.5964×10^{-4}	1.2505×10^{-4}
h_3	-7.3187×10^{-8}	1.7017×10^{-8}

Table 2. The computed discretized and continuous arc lengths of a dipole field line (s_{gc}, s_{dc}, ℓ) in [km] and their relative errors $(\mathcal{E}_{gc/dc}, \mathcal{E}_{gc/\ell}, \mathcal{E}_{dc/\ell})$.

s_{gc}	s_{dc}	ℓ	$\mathcal{E}_{gc/dc}$	$\mathcal{E}_{gc/\ell}$	$\mathcal{E}_{dc/\ell}$
21 480.2394	21 480.2403	21 480.5278	-4.0869×10^{-8}	-1.3424×10^{-5}	-1.3383×10^{-5}

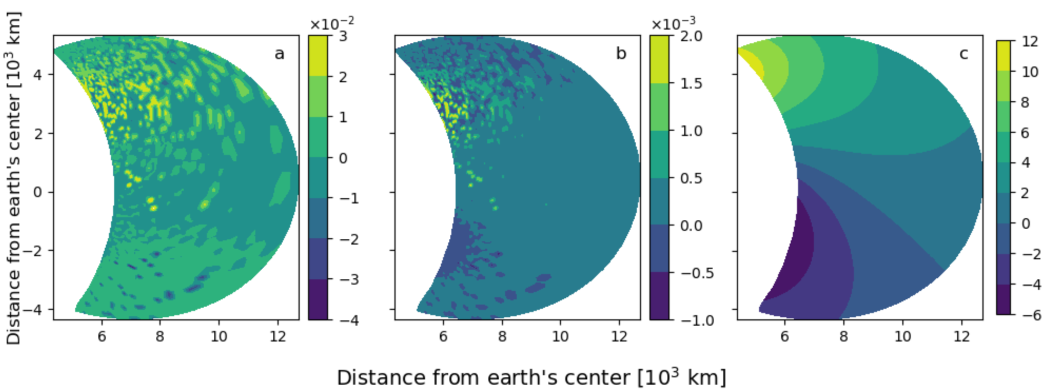

Figure 2. The departures from 90° of the angles between the basis vectors: a) β_{12}, b) β_{13}, and c) β_{23}.

Since both $\mathbf{e}^2$ and $\mathbf{e}^3$ are perpendicular to $\mathbf{e}^1$, their cross product $\mathbf{e}^2 \times \mathbf{e}^3$ would be *parallel* to $\mathbf{e}^1$, hence also the magnetic field $\mathbf{B}$. Thus, we can write the magnetic field in the following way:

$$\mathbf{B} \equiv -g_m \nabla \mu_m = -c \nabla \chi_m \times \nabla \phi_m,$$

Figure 3 plots the ratio $r = c/g_m$, and the distribution of grid points along the field lines. It shows that the ratio follows the field lines, i.e., it is constant along each field line. Thus, the constant c depends on the field line, but is fixed for each field line. Therefore, χ_m and ϕ_m in the general coordinates, as the corresponding χ and ϕ in the dipole coordinates, are also Euler potentials. Note that, for a dipole field, the ratio $r = c/g_m$ would be a constant 1 everywhere.

The Euler potential, as a potential function, is not uniquely defined. Several different ways to define and construct the Euler potentials have been proposed and used, e.g., Stern (1967, 1976, 1994); Ho et al. (1997); Peymirat & Fontaine (1999); Wolf et al. (2006); Rankin et al. (2006). Sometimes the computational procedures can get quite complicated. So depending on the applications, the simple approach here may be sometimes preferable.

Table 3. The minima and maxima of the departures from 90° of the angles between the basis vectors.

β_{ij}	Min	Max
β_{12}	-4.4080×10^{-2}	4.1725×10^{-2}
β_{13}	-9.3284×10^{-4}	2.5954×10^{-3}
β_{23}	-5.8443	11.0157

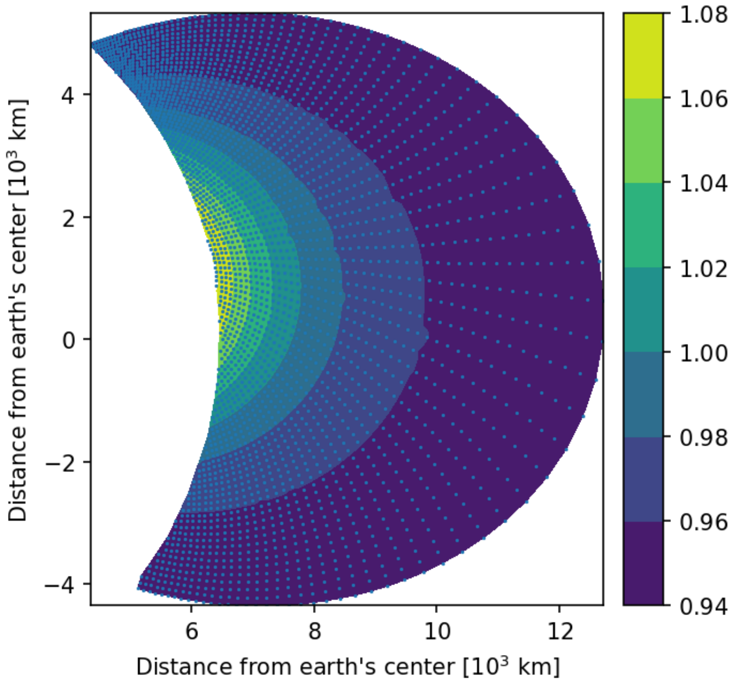

Figure 3. The ratio $r = c/g_m$: $r_{\min} = 0.9521, r_{\max} = 1.0824$. The dots are grid points.

5 Summary

A general curvilinear coordinate system is proposed for ionosphere-plasmasphere modeling. This magnetic field-line-following curvilinear coordinate system becomes a usual definition of dipole coordinate system when the magnetic field becomes the dipole.

High-order ordinary differential equation (ODE) solver is used to solve the magnetic field line equations for the general magnetic field of the IGRF model. The numerical accuracy and consistency of the implementation is validated against the analytical results in the case of a dipole magnetic field. The symmetry of the dipole field can also be used to check the consistency and accuracy of implementation: any loss of symmetry may indicate loss of accuracy or lack of consistency in the implementation of numerical algorithms.

The general coordinate system is also the Euler potential or Clebsch-type coordinate system. There are infinite choices of Euler potentials for coordinate variables, and it can become complicated sometimes. So depending on the applications, the simple approach used here may be preferred in some cases.

The general curvilinear magnetic field-line-following coordinate system proposed here is developed and implemented while developing a new ionosphere-plasmasphere model. It is also an attempt to put the field-line-following coordinate system on a more rigorous or conventional mathematical framework. Applications of the general coordinates will be presented in a separate paper on ionosphere-plasmasphere modeling.

Appendix A The IGRF magnetic field

The Earth's main magnetic field can be best described by the International Geomagnetic Reference Field (IGRF) (Thébault et al. (2015)). The magnetic potential Φ of IGRF is approximated by the truncated series, written in the *geographic spherical coordinates* (r, θ, ϕ), as follows:

$$\Phi(r, \theta, \phi, t) = a \sum_{n=1}^{N} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} [g_n^m(t) \cos(m\phi) + h_n^m(t) \sin(m\phi)] P_n^m(\cos\theta), \tag{A1}$$

where r is the radial distance from the center of the Earth, $a = 6371.2$ km is the geomagnetic conventional Earth's mean reference spherical radius, θ is the geocentric colatitude, and ϕ the east longitude. The functions $P_n^m(\cos\theta)$ are the Schmidt quasi-normalized associated Legendre functions of degree n and order m. The order of approximation is truncated to $N = 10$ for epochs up to 1995.0 and $N = 13$ from epoch 2000. In numerical computation of this study, the Schmidt quasi-normalized associated Legendre functions are evaluated using the SHTOOLS (Wieczorek & Meschede (2018)).

The lowest-order approximation, by setting $n = 1$ and $m = 0$ in the truncated series of Eq. (A1), defines an *axial-centered dipole field*:

$$\Phi(r, \theta) = a \left(\frac{a}{r}\right)^2 g_1^0 \cos\theta \equiv g_m \left(\frac{a}{r}\right)^2 \cos\theta, \tag{A2}$$

where we have defined

$$g_m \equiv a g_1^0 \tag{A3}$$

and may simply be called the *dipole moment*.

The three components of the magnetic field $\mathbf{B} = -\nabla\Phi$ in the spherical coordinates are computed by

$$B_r = \sum_{n=1}^{N} (n+1) \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} [g_n^m \cos(m\phi) + h_n^m \sin(m\phi)] P_n^m(\cos\theta), \tag{A4a}$$

$$B_\theta = \sin\theta \sum_{n=1}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} [g_n^m \cos(m\phi) + h_n^m \sin(m\phi)] \frac{\partial P_n^m(x)}{\partial x}, \tag{A4b}$$

$$B_\phi = \frac{1}{\sin\theta} \sum_{n=1}^{N} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} m [g_n^m \sin(m\phi) - h_n^m \cos(m\phi)] P_n^m(\cos\theta), \tag{A4c}$$

where we have used

$$\frac{\partial P_n^m(\cos\theta)}{\partial\theta} = -\sin\theta \frac{\partial P_n^m(x)}{\partial x}$$

by chain rule with $x = \cos\theta$.

Appendix B Numerical solution of the differential equations for the magnetic field line

B1 The differential equations for the dipole magnetic field line

The differential equations for a dipole field line can be written in the spherical coordinates as

$$\frac{dr}{r d\theta} = \frac{B_r}{B_\theta} = \frac{2\cos\theta}{\sin\theta}. \tag{B1}$$

An analytical expression for the arc length of a dipole field line can be obtained, see e.g. (Walt, 1994, pp. 30-31). Integrating the field line equation (B1) gives

$$r = r_A \sin^2 \theta,$$

where r_A is the value of r at apex $\theta = \pi/2$. The arc distance element $d\ell$ along a field line

$$d\ell = \sqrt{(dr)^2 + (rd\theta)^2},$$

can be integrated analytically as follows. Differentiating $r = r_A \sin^2 \theta$ gives

$$dr = 2r_A \sin\theta \cos\theta d\theta.$$

Thus,

$$\begin{aligned} d\ell &= \sqrt{4r_A^2 \sin^2\theta \cos^2\theta + r_A^2 \sin^4\theta} d\theta \\ &= r_A \sqrt{1 + 3\cos^2\theta} \sin\theta d\theta. \end{aligned}$$

Let $x = \cos\theta$, integrating from $x = \cos\theta$ to the equator $x = \cos(\pi/2) = 0$, we get an analytical expression for the arc length of a dipole field line as follows:

$$\ell = \frac{r_A}{2}\left[x\sqrt{1+3x^2} + \frac{1}{\sqrt{3}}\ln(\sqrt{1+3x^2} + \sqrt{3}x)\right]. \tag{B2}$$

Note that we use ℓ here to distinguish it from the arc length s calculated from the discretized form Eq. (6).

B2 The differential equations for the general magnetic field line

For a general magnetic field such as IGRF, the differential equations for a field line can be written as

$$\frac{\delta s}{B} = \frac{\delta r}{B_r} = \frac{r\delta\theta}{B_\theta} = \frac{r\sin\theta\delta\phi}{B_\phi}, \tag{B3}$$

which can be solved for field line (tracing) as follows

$$r^{n+1} = r^n + \hat{\tau}\left(\frac{B_r}{B}\right)^n (\delta s)^n, \tag{B4a}$$

$$\theta^{n+1} = \theta^n + \hat{\tau}\frac{1}{r^n}\left(\frac{B_\theta}{B}\right)^n (\delta s)^n, \tag{B4b}$$

$$\phi^{n+1} = \phi^n + \hat{\tau}\frac{1}{r^n \sin\theta^n}\left(\frac{B_\phi}{B}\right)^n (\delta s)^n, \tag{B4c}$$

where superscripts n and $n+1$ are position indices; $\hat{\tau} = \hat{\mathbf{s}} \cdot \hat{\mathbf{b}} = \pm 1$, with $+$ (or $-$) sign depending on whether $\hat{\mathbf{s}}$ moving/tracing in the same (or opposite) direction of $\hat{\mathbf{b}}$ (the unit vector of the magnetic field $\mathbf{B}$).

B3 Numerical solution of the general magnetic field line equations

In many applications, such as grid generation described in C2, accurate and efficient high-order ordinary differential equation (ODE) solvers are needed to solve the magnetic field line tracing equations (B4) numerically. In this study, we use the higher-order embedded method, the Runge-Kutta-Fehlberg (RKF45) method. RKF45 is a method of order $O(h^4)$ with an error estimator of order $O(h^5)$, which automatically determines the step-size to achieve the pre-defined accuracy. The RKF45 solver implemented in package rksuite_90 (Brankin & Gladwell, 1997) is used in this study. Highly accurate results are obtained, as shown in section 4.1 on comparing the numerically computed and the analytically derived results in the case of the dipole field.

Appendix C Grid generation for ionosphere-plasmasphere modeling

C1 Grid generation for dipole coordinates

For the dipole coordinates (DC), the grid uniform in (μ, χ, ϕ) with constant $(d\mu, d\chi, d\phi)$ can be generated as follows:

1) Choose two magnetic colatitudes θ_1 and θ_2, where the outmost and innermost field lines intersect with the earth's surface, and a longitudes ϕ.
2) From the constancy of $\chi = \sin^2\theta/r$ along each field line, find the corresponding radial distances at the magnetic equator, i.e., at apex where $\theta = \pi/2$. These are given by $r_{A1} = 1/\sin^2\theta_1$ and $r_{A2} = 1/\sin^2\theta_2$. Grid increment $d\chi$ is determined by dividing $\chi_1 = 1/r_{A1}$ and $\chi_2 = 1/r_{A2}$ into a predefined number of field lines in the meridional plane (ϕ = const.).
3) Grid distribution along the field lines: grid increment $d\mu$ is determined by dividing the potential at the earth's surface $\mu_0 = -\cos(\theta_0)$ and the potential at the apex $\mu_A = 0$ of the outmost field line into a predefined number of points. Grid along the inner field lines is generated using the same grid increment $d\mu$, starting from the apex. The foot points for each field line are chosen to be the lowest points above the spherical Earth's surface that satisfy e.g., $r \geqslant a + 90$ [km].
4) Compute magnetic flux density, basis vectors, metric terms, scale factors, and the arc length etc.

Analytical expressions of the inverse transformation from the dipole coordinates (μ, χ, ϕ) to the spherical coordinates (r, θ, ϕ) are given by Kageyama et al. (2006). These are used in the grid generation for the dipole coordinates.

Note that for the dipole coordinates, $\theta_g = \theta_m$, where the subscripts g and m indicate geographic/geodetic and magnetic colatitudes, respectively. So we don not differentiate between the two in the dipole coordinates.

C2 Grid generation for general coordinates

As in defining the general coordinates, which becomes a usual definition of dipole coordinates the field becomes a dipole, grid generation for the general coordinates can be done in a similar way as that for the dipole coordinates.

For the general coordinates (GC), the grid uniform in (μ_m, χ_m, ϕ_m) with constant $(d\mu_m, d\chi_m, d\phi_m)$ can be generated as follows::

1) Choose two magnetic colatitudes θ_{m1} and θ_{m2}, where the outmost and innermost field lines intersect with the earth's surface, and a longitude $\phi_m = \phi_{gA}$ at apex.
2) From the constancy of $\chi_m = \sin^2\theta_m/r$ along each field line, find the corresponding radial distances at the magnetic equator, i.e., at apex where $\theta_m = \pi/2$. These are given by $r_{A1} = 1/\sin^2\theta_{m1}$ and $r_{A2} = 1/\sin^2\theta_{m2}$. Grid increment $d\chi_m$ is determined by dividing $\chi_{m1} = 1/r_{A1}$ and $\chi_{m2} = 1/r_{A2}$ into a predefined number of field lines n the meridional plane (ϕ_m = const.).
3) Grid distribution along the field lines:
   a) Find the geographic/geodetic colatitude at apex: given the geographic coordinates at apex (r_{gA}, ϕ_{gA}) and a first-guess of θ'_{gA}, find θ_{gA} (θ_g at apex) using the Newton-Raphson method; and compute the magnetic potential Φ_{mA} at apex. *The apex is defined by $B_r = 0$, thus a turning point.*
   b) Tracing down from apex to locate where the field line cross the Earth's surface on both hemispheres: $(r_{g0}, \theta_{g0}, \phi_{g0})$, and compute the magnetic potential Φ_{m0}.
   c) Grid increment along the field lines is determined by dividing the potential at the earth's surface Φ_{m0} and the potential at the apex Φ_{mA} of the outmost field line into a predefined number of points. Grid along the inner field lines is generated using the same grid increment, starting from the apex. The foot points

for each field line are chosen to be the lowest points above the spherical Earth's surface that satisfy e.g., $r \geqslant a + 90$ [km].

4) Compute magnetic flux density, basis vectors, metric terms, scale factors, and the arc length etc.

References

Bailey, G. J., Sellek, R., & Rippeth, Y. (1993). A modelling study of the equatorial topside ionosphere. *Ann. Geophys.*, *11*, 263–272.

Brankin, R. W., & Gladwell, I. (1997). Algorithm 771: rksuite_90: Fortran 90 Software for Ordinary Differential Equation Initial-Value Problems. *ACM Trans. Math. Softw.*, *23*(3), 402–415. doi: 10.1145/275323.275328

D'haeseleer, W. D., Hitchon, W. N. G., Callen, J. D., & Shohet, J. L. (1991). *Flux Coordinates and Magnetic Field Structure.* Springer-Verlag.

Emmert, J. T., Richmond, A. D., & Drob, D. P. (2010). A computationally compact representation of Magnetic-Apex and Quasi-Dipole coordinates with smooth base vectors. *J. Geophys. Res. Sp. Phys.*, *115*(A8). doi: 10.1029/2010JA015326

Ho, C. W., Huang, T. S., & Gao, S. (1997). Contributions of the high-degree multipoles of Neptune's magnetic field: An Euler potentials approach. *J. Geophys. Res. A Sp. Phys.*, *102*(A11), 24393–24401. doi: 10.1029/97JA01649

Huba, J. D., Joyce, G., & Fedder, J. A. (2000). Sami2 is Another Model of the Ionosphere (SAMI2): A new low-latitude ionosphere model. *J. Geophys. Res.*, *105*.

Kageyama, A., Sugiyama, T., Watanabe, K., & Sato, T. (2006). A note on the dipole coordinates. *Comput. Geosci.*, *32*(2), 265–269. doi: 10.1016/j.cageo.2005.06.006

Laundal, K. M., & Richmond, A. D. (2017). Magnetic Coordinate Systems. *Space Sci. Rev.*, *206*(1-4), 27–59. doi: 10.1007/s11214-016-0275-y

Peymirat, C., & Fontaine, D. (1999). A numerical method to compute Euler potentials for non dipolar magnetic fields. *Ann. Geophys.*, *17*(3), 328–337. doi: 10.1007/s00585-999-0328-6

Rankin, R., Kabin, K., & Marchand, R. (2006). Alfvénic field line resonances in arbitrary magnetic field topology. *Adv. Sp. Res.*, *38*(8), 1720–1729. doi: 10.1016/j.asr.2005.09.034

Richmond, A. D. (1995). Ionospheric electrodynamics using magnetic apex coordinates. *J. Geomagn. Geoelectr.*, *47*(2), 191–212. doi: 10.5636/jgg.47.191

Stern, D. P. (1967, aug). Geomagnetic Euler potentials. *J. Geophys. Res.*, *72*(15), 3995–4005. doi: 10.1029/JZ072i015p03995

Stern, D. P. (1970). Euler Potentials. *Am. J. Phys.*, *38*(4), 494–501. doi: 10.1119/1.1976373

Stern, D. P. (1976). Representation of magnetic fields in space. *Rev. Geophys.*, *14*(2), 199–214. doi: 10.1029/RG014i002p00199

Stern, D. P. (1994). Euler potentials of current-free fields expressed in spherical harmonics. *J. Geophys. Res.*, *99*(A2), 2443. doi: 10.1029/93ja03168

Thébault, E., Finlay, C. C., Beggan, C. D., Alken, P., Aubert, J., Barrois, O., ... Zvereva, T. (2015). International Geomagnetic Reference Field: the 12th generation. *Earth, Planets Sp.*, *67*(1), 79. doi: 10.1186/s40623-015-0228-9

VanZandt, T. E., Clark, W. L., & Warnock, J. M. (1972). Magnetic apex coordinates: A magnetic coordinate system for the ionospheric F2 layer. *J. Geophys. Res.*, *77*(13), 2406–2411. doi: 10.1029/JA077i013p02406

Walt, M. (1994). *Introduction to Geomagnetically Trapped Radiation.* Cambridge University Press.

Wieczorek, M. A., & Meschede, M. (2018). SHTools: Tools for Working with Spherical Harmonics. *Geochemistry, Geophys. Geosystems*, *19*(8), 2574–2592. doi: 10.1029/2018GC007529

Wolf, R. A., Spiro, R. W., Sazykin, S., Toffoletto, F. R., Le Sager, P., & Huang, T.-S. (2006). Use of Euler potentials for describing magnetosphere-ionosphere coupling. *J. Geophys. Res. Sp. Phys.*, *111*(A7), A07315. doi: 10.1029/2005JA011558