Non-singular calculation of geomagnetic vectors and geomagnetic gradient tensors

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Abstract

Some spherical harmonic expressions (SHEs) of gravitational and geomagnetic field elements will become infinite when computation point approaches polar regions, as the sine function of the geocentric co-latitude contained in the denominator tends to be zero. Currently, this singularity problem has been solved for gravitational field case, however, it remains unsolved for geomagnetic vectors (GVs) and geomagnetic gradient tensors (GGTs). The reason is that the latter use Schmidt semi-normalized associated Legendre function (SNALF), which is different from fully-normalized associated Legendre function (FNALF) used in the former. To overcome this singularity problem, we derive new non-singular expressions of the first- and second-order derivatives of Schmidt SNALF, and the corresponding two kinds of spherical harmonic polynomials. When the novel expressions are applied to the traditional formulae of GVs and GGTs, more practical expressions of GVs and GGTs with non-singularity are formulated by refining the cases that the order m equals 0, 1, 2 and other values. Furthermore, to provide flexible calculation strategies for Schmidt SNALF, we derive four kinds of recursive formulae, including the standard forward row recursion (SFRR), the standard forward column recursion (SFCR), the cross degree and order recursion (CDOR), and the Belikov recursion (BR). Besides, we demonstrate the effectiveness of the new derived non-singular expressions of GVs and GGTs and analyze the computation speed and stability of the four recursive formulae of Schmidt SNALF by extensive numerical experiments. Results achieve significant improvements in solving the singularity problem of the SHEs of GVs and GGTs.

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Key Points:

- More practical expressions of geomagnetic vectors and geomagnetic gradient tensors with non-singularity are formulated.
- Four kinds of recursive formulae of Schmidt semi-normalized associated Legendre function are derived.
- We achieve significant improvements in solving singularity problem in geomagnetic vectors and geomagnetic gradient tensors at poles.

Abstract Some spherical harmonic expressions (SHEs) of gravitational and geomagnetic field elements will become infinite when computation point approaches polar regions, as the sine function of the geocentric co-latitude contained in the denominator tends to be zero. Currently, this singularity problem has been solved for gravitational field case, however, it remains unsolved for geomagnetic vectors (GVs) and geomagnetic gradient tensors (GGTs). The reason is that the latter use Schmidt semi-normalized associated Legendre function (SNALF), which is different from fully-normalized associated Legendre function (FNALF) used in the former. To overcome this singularity problem, we derive new non-singular expressions of the first- and second-order derivatives of Schmidt SNALF, and the corresponding two kinds of spherical harmonic polynomials. When the novel expressions are applied to the traditional formulae of GVs and GGTs, more practical expressions of GVs and GGTs with non-singularity are formulated by refining the cases that the order m equals 0, 1, 2 and other values. Furthermore, to provide flexible calculation strategies for Schmidt SNALF, we derive four kinds of recursive formulae, including the standard forward row recursion (SFRR), the standard forward column recursion (SFCR), the cross degree and order recursion (CDOR), and the Belikov recursion (BR). Besides, we demonstrate the effectiveness of the new derived non-singular expressions of GVs and GGTs and analyze the computation speed and stability of the four recursive formulae of Schmidt SNALF by extensive numerical experiments. Results achieve significant improvements in solving the singularity problem of the SHEs of GVs and GGTs.

Key words Geomagnetic Vectors (GVs); Geomagnetic Gradient Tensors (GGTs); Non-singular; Spherical Harmonic Function; Legendre Function; Polar Region

Plain Language Summary According to the spherical harmonic expressions (SHEs) of geomagnetic field elements (such as geomagnetic potential, geomagnetic vectors (GVs), and geomagnetic gradient tensors (GGTs)), some GVs and GGTs become infinite at poles when sine function of geocentric co-latitude in the denominator is equal to zero, which is called the singularity problem. For satellite and aviation magnetic measurement, it is necessary to solve the singularity problem of SHEs of the geomagnetic field elements in the polar regions. Therefore, we formulate new non-singular expressions of GVs and GGTs based on a linear combination of Schmidt semi-normalized associated Legendre function (SNALF), which achieves significant improvements. In addition, four recursive formulae of Schmidt SNALF are also derived, and a flexible calculation strategies for Schmidt SNALF are presented. The research results can be applied to data processing and modeling of airbore and satellite measurements of GVs and GGTs in the polar regions.

1 Introduction

Spherical harmonic functions are generally developed to represent gravitational field elements (such as disturbing potential, disturbing gravity vectors, and disturbing gravity gradient tensors), and geomagnetic field elements (such as geomagnetic potential, geomagnetic vectors (GVs), and geomagnetic gradient tensors (GGTs)), because it can simplify calculations. However, according to the spherical harmonic expressions (SHEs) of gravitational and geomagnetic field elements, some GVs and GGTs become infinite at poles when the sine function of geocentric co-latitude in the denominator is equal to zero, which is called the singularity problem. In fact, the gravitational and geomagnetic field elements should be finite. Due to limited sphere of activity in early days, it is not necessary to solve the singularity problem, and people usually take flexible measures to avoid it in practice. For example, gravitational and geomagnetic field elements are directly expressed in Cartesian coordinates. However, the complicated formulae and parameters tend to result in unstable and inaccurate calculations. With the development of satellite and aviation gravimetry and magnetic measurement, the sphere of human activities are expanded across the globe, which makes it necessary to solve the singularity problem of SHEs of the gravitational and geomagnetic field elements.

Some previous efforts have solved the singularity problem of SHEs of gravitational field elements (Hotine and Morrison, 1969; Ilk, 1983; Balmino et al., 1990; Bettadpur, 1995; Petrovskaya and Vershkov, 2006, 2007, 2008; Casotto and Fantino, 2007; Eshagh, 2008, 2009; Eshagh and Sjöberg, 2009; Wan, 2011; Liu et al., 2010, 2013; Zhu et al., 2017), whereas the same problem still exists in GVs and GGTs. The reason is that the gravitational field elements are expressed by fully-normalized associated Legendre function (FNALF) (Chen et al., 2006;

Jekeli and Lee, 2007; Fantino and Casotto, 2009; Hirt et al., 2010; Rummel et al., 2011; Pail et al., 2011; Liu et al., 2012; Fukushima, 2012a, 2012b; Pavlis et al., 2012; Wan and Yu, 2013; Šprlák and Novák, 2017), while the GVs and GGTs are expressed by Schmidt semi-normalized associated Legendre function (SNALF) (Malin and Pocock, 1969; Barraclough, 1974; Benton et al., 1982; Quinn et al., 1986; Blakely, 1995; Ravat et al., 1995; Shao et al., 1999; Chambodut et al., 2005; Hemant and Maus, 2005; Wardinski and Holme, 2006; Kim et al., 2007; Huang et al., 2011; Kotsiaros and Olsen, 2012; Du et al., 2015; Liu et al., 2019), whose recursive formulae is different from that of FNALF.

Although non-singular formulae of GVs and GGTs have been derived by Du et al. (2015), there are still some drawbacks. For example, the non-singular formula of the first-order derivative of Schmidt SNALF is incorrect when the order m is equal to 0, and the formula of the second-order derivative of Schmidt SNALF is also incorrect when the order m is equal to 0 or 1. Hence, it can be deduced that those non-singular formulae lead to incorrect computation of GVs and GGTs, especially when order m is equal to 0 or 1. The goal of this paper is to derive new non-singular expressions of GVs and GGTs, which can be used in geomagnetism, geophysics, geodesy, and other related disciplines. In addition, considering the efficiency and accuracy of the calculation of Schmidt SNALF, four kinds of recursive formulae are introduced, i.e., the standard forward row recursion (SFRR), the standard forward column recursion (SFCR), the cross degree and order recursion (CDOR), and the Belikov recursion (BR). Numerical experiments are performed to demonstrate the conclusion.

The article is organized as follows. Section 2 formulates the singularity problem. Section 3 derives new non-singular expressions for both GVs and GGTs. Section 4 introduces four kinds of recursive formulae of Schmidt SNALF. Section 5 demonstrates the effectiveness of newly derived non-singular expressions, and analyzes the speed and stability through extensive numerical experiments. Section 6 summarizes the main contributions and draws the conclusion.

2 Statement of the problem

The traditional SHEs of GVs are expressed as:

$$B_z = -\sum_{n=1}^{N}(n+1)\left(\frac{R}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m \cos m\lambda + h_n^m \sin m\lambda\right]\bar{P}_n^m(\cos\theta) \tag{1}$$

$$B_x = \sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m \cos m\lambda + h_n^m \sin m\lambda\right]\frac{d\bar{P}_n^m(\cos\theta)}{d\theta} \tag{2}$$

$$B_y = \sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m \sin m\lambda - h_n^m \cos m\lambda\right]\frac{m\bar{P}_n^m(\cos\theta)}{\sin\theta} \tag{3}$$

The traditional SHEs of GGTs are expressed as:

$$B_{zz} = \frac{1}{R}\sum_{n=1}^{N}(n+1)(n+2)\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \cos m\lambda + h_n^m \sin m\lambda\right]\bar{P}_n^m(\cos\theta) \tag{4}$$

$$B_{xx} = \frac{1}{R}\sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \cos m\lambda + h_n^m \sin m\lambda\right]\left[\frac{d^2\bar{P}_n^m(\cos\theta)}{d\theta^2} - (n+1)\bar{P}_n^m(\cos\theta)\right] \tag{5}$$

$$\begin{aligned} B_{yy} = {} & \frac{1}{R}\sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \cos m\lambda + h_n^m \sin m\lambda\right]\cdot \\ & \left[\frac{d\bar{P}_n^m(\cos\theta)}{d\theta}\frac{\cos\theta}{\sin\theta} - \frac{m^2\bar{P}_n^m(\cos\theta)}{\sin^2\theta} - (n+1)\bar{P}_n^m(\cos\theta)\right] \end{aligned} \tag{6}$$

$$B_{xy} = -\frac{1}{R}\sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \sin m\lambda - h_n^m \cos m\lambda\right]\left[\frac{m\cos\theta}{\sin^2\theta}\bar{P}_n^m(\cos\theta) - \frac{m}{\sin\theta}\frac{d\bar{P}_n^m(\cos\theta)}{d\theta}\right] \tag{7}$$

$$B_{xz} = -\frac{1}{R}\sum_{n=1}^{N}(n+2)\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \cos m\lambda + h_n^m \sin m\lambda\right]\frac{d\bar{P}_n^m(\cos\theta)}{d\theta} \tag{8}$$

$$B_{yz} = -\frac{1}{R}\sum_{n=1}^{N}(n+2)\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \sin m\lambda - h_n^m \cos m\lambda\right]\frac{m\bar{P}_n^m(\cos\theta)}{\sin\theta} \tag{9}$$

In Equations ~, (x, y, z) are the coordinates in the local-north-oriented reference frame (LNORF), where the z axis points downward in the geocentric radial direction, the x axis points to the north, and the y axis points to the east (that is, a right-handed system). B_x, B_y, B_z denote the north, east and vertical component of GVs, respectively. B_{xx}, B_{yy}, B_{zz}, B_{xy}, B_{xz} and B_{yz} denote six components of GGTs. R=6371.2 km is the average radius of the Earth; r is the geocentric radius; θ is the geocentric co-latitude; λ is the geocentric longitude; n and m are the degree and order of spherical harmonic coefficients, respectively; N is the truncation order; g_n^m, h_n^m are the Gauss spherical harmonic coefficients of the internal field.

$\bar{P}_n^m(\cos\theta)$ is the Schmidt SNALF, which can be expressed as

$$\bar{P}_n^m(\cos\theta) = \sqrt{(2-\delta_0^m)\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta) \tag{10}$$

where δ_0^m is the Kroneker symbol

$$\delta_0^m = \begin{cases} 1 & m = 0 \\ 0 & m > 0 \end{cases} \tag{11}$$

$P_n^m(\cos\theta)$ is the Legendre polynomial, which can be expressed as

$$P_n^m(\cos\theta) = \sin^m\theta\frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m} = -\frac{1}{2^n n!}\sin^m\theta\frac{d^{n+m}\sin^{2n}\theta}{d(\cos\theta)^{n+m}} \tag{12}$$

$d\bar{P}_n^m(\cos\theta)/d\theta$ is the first-order derivative of Schmidt SNALF, its traditional recursive formula is (Heiskanen and Moritz, 1967; Holmes and Featherstone, 2002a, 2002b; Fukushima, 2012a)

$$\frac{d\overline{P}_n^m(\cos\theta)}{d\theta} = m\frac{\cos\theta}{\sin\theta}\overline{P}_n^m(\cos\theta) - \sqrt{\frac{(n+m+1)(n-m)}{1+\delta_0^m}}\overline{P}_n^{m+1}(\cos\theta) \tag{13}$$

To continue the derivation of Equation , the traditional recursive formula of the second-order derivative of Schmidt SNALF, $d^2\overline{P}_n^m(\cos\theta)/d\theta^2$ can be obtained as

$$\begin{aligned}\frac{d^2\overline{P}_n^m(\cos\theta)}{d\theta^2} &= \left[m^2\frac{\cos^2\theta}{\sin^2\theta} - \frac{m}{\sin^2\theta} - (n-m)(n+m+1)\right]\overline{P}_n^m(\cos\theta) + \\ &\sqrt{\frac{1}{1+\delta_0^m}}\frac{\cos\theta}{\sin\theta}\sqrt{(n-m)(n+m+1)}\,\overline{P}_n^{m+1}(\cos\theta)\end{aligned} \tag{14}$$

From Equations and , it is straightforward to see that the first- and second-order derivatives of Schmidt SNALF become infinite when the computation point is approaching the poles, because the sinθ in the denominator equals zero. Since the B_x, B_{xx}, B_{yy}, B_{xy} and B_{xz} contain the first- and second-order derivatives of Schmidt SNALF, as shown in Equations , and -, the singularity problem also exists in the calculation of above GVs and GGTs.

Equations , , , and contain two kinds of spherical harmonic polynomials, namely $m\overline{P}_n^m(\cos\theta)/\sin\theta$ and $m^2\overline{P}_n^m(\cos\theta)/\sin^2\theta$, which will be infinite at the poles when the sinθ in the denominator equals zero. It means that the singularity problem will also occur in the calculation of B_y, B_{yy}, B_{xy} and B_{yz}.

Based on the above analysis, we know that some GVs and GGTs still have the singularity problem in the polar regions. Although Du et al. (2015) provided non-singular expressions of $d\overline{P}_n^m(\cos\theta)/d\theta$, $d^2\overline{P}_n^m(\cos\theta)/d\theta^2$, $\overline{P}_n^m(\cos\theta)/\sin\theta$, $\overline{P}_n^m(\cos\theta)/\sin^2\theta$, $d\overline{P}_n^m(\cos\theta)/(\sin\theta d\theta)$, etc., the non-singular expressions of the first- and the second-order derivatives of Schmidt SNALF are incorrect when the order m is equal to 0 or 1, which leads to incorrect non-singular expressions of some GVs and GGTs. In order to solve this problem, we derive new non-singular expressions of the first- and second-order derivatives of Schmidt SNALF and its two kinds of spherical harmonic polynomials, and formulate more practical non-singular expressions of GVs and GGTs.

3 New non-singular expressions of GVs and GGTs

3.1 Development of the non-singular first-order derivative of Schmidt SNALF

When $m \geqslant 1$, according to Equation , we can obtain

$$P_n^m(\cos\theta) = \sin^m\theta\frac{d^m P_n(\cos\theta)}{d(\cos\theta)^m} = \sin^m\theta P_n^{(m)}(\cos\theta) \tag{15}$$

For equation (15), first-order derivative with respect to θ reads

$$\frac{dP_n^m(\cos\theta)}{d\theta} = -\sin^{m+1}\theta P_n^{(m+1)}(\cos\theta) + m\sin^{m-1}\theta\cos\theta P_n^{(m)}(\cos\theta) \tag{16}$$

According to the differential equation of $P_n(x)$ (Heiskanen and Moritz, 1967; Moritz, 1980; Bernhard and Moritz, 2006),

$$(1-x^2)P_n^{(m+2)}(x) - 2(m+1)xP_n^{(m+1)}(x) + (n-m)(n+m+1)P_n^{(m)}(x) = 0 \tag{17}$$

Replacing x in Equation with $\cos\theta$, the resulting equation can be expressed as

$$\sin^2\theta P_n^{(m+2)}(\cos\theta) - 2(m+1)\cos\theta P_n^{(m+1)}(\cos\theta) + (n-m)(n+m+1)P_n^{(m)}(\cos\theta) = 0 \tag{18}$$

We obtain

$$P_n^{(m)}(\cos\theta) = \frac{2(m+1)\cos\theta P_n^{(m+1)}(\cos\theta) - \sin^2\theta P_n^{(m+2)}(\cos\theta)}{(n-m)(n+m+1)} \tag{19}$$

Substituting Equation into Equation , we obtain

$$\begin{aligned}\frac{dP_n^m(\cos\theta)}{d\theta} = &-\sin^{m+1}\theta P_n^{(m+1)}(\cos\theta) \\ &+ \frac{2m(m+1)\sin^{m-1}\theta\cos^2\theta P_n^{(m+1)}(\cos\theta) - m\sin^{m+1}\theta\cos\theta P_n^{(m+2)}(\cos\theta)}{(n-m)(n+m+1)}\end{aligned} \tag{20}$$

According to Equation , we find below relations

$$P_n^{(m+1)}(\cos\theta) = \frac{P_n^{m+1}(\cos\theta)}{\sin^{m+1}\theta} \tag{21}$$

$$P_n^{(m+2)}(\cos\theta) = \frac{P_n^{m+2}(\cos\theta)}{\sin^{m+2}\theta} \tag{22}$$

Substituting Equations and into Equation , we obtain:

$$\frac{dP_n^m(\cos\theta)}{d\theta} = -P_n^{m+1}(\cos\theta) + \frac{2m(m+1)\cos^2\theta P_n^{m+1}(\cos\theta) - m\sin\theta\cos\theta P_n^{m+2}(\cos\theta)}{(n-m)(n+m+1)\sin^2\theta} \tag{23}$$

According to the the recursive formulae of the Legendre polynomial of the SFRR (Colombo, 1981; Holmes and Featherstone, 2002a, 2002b)

$$P_n^m(\cos\theta) = \begin{cases} \dfrac{1}{(n-m)(n+m+1)}\left[\dfrac{2(m+1)\cos\theta}{\sin\theta}P_n^{m+1}(\cos\theta) - P_n^{m+2}(\cos\theta)\right] & n > m \\ (2m-1)\sin\theta P_{m-1}^{m-1}(\cos\theta) & n = m(m>0) \end{cases} \tag{24}$$

It can be derived that

$$P_n^{m+2}(\cos\theta) = 2(m+1)\frac{\cos\theta}{\sin\theta}P_n^{m+1}(\cos\theta) - (n-m)(n+m+1)P_n^m(\cos\theta) \tag{25}$$

Substituting Equation into Equation , we can obtain

$$\frac{dP_n^m(\cos\theta)}{d\theta} = -P_n^{m+1}(\cos\theta) + \frac{m\cos\theta}{\sin\theta}P_n^m(\cos\theta) \tag{26}$$

According to the Equation , we can write

$$P_n^{m+1}(\cos\theta) = 2m\frac{\cos\theta}{\sin\theta}P_n^m(\cos\theta) - (n-m+1)(n+m)P_n^{m-1}(\cos\theta) \tag{27}$$

From Equation (27), we can obtain:

$$m\frac{\cos\theta}{\sin\theta}P_n^m(\cos\theta) = \frac{1}{2}P_n^{m+1}(\cos\theta) + \frac{1}{2}(n-m+1)(n+m)P_n^{m-1}(\cos\theta) \tag{28}$$

Substituting Equation into Equation , the non-singular formula of the first-order derivative of the Legendre polynomial when $m \geqslant 1$ can be expressed as

$$\frac{dP_n^m(\cos\theta)}{d\theta} = -\frac{1}{2}P_n^{m+1}(\cos\theta) + \frac{1}{2}(n-m+1)(n+m)P_n^{m-1}(\cos\theta) \tag{29}$$

According to the Equation , the non-singular formula of the first-order derivative of Legendre polynomial when m=0 can be expressed as

$$\frac{dP_n^0(\cos\theta)}{d\theta} = -P_n^1(\cos\theta) \tag{30}$$

According to Equation , Equations and are Schmidt semi-standardized and the non-singular formulae of the first-order derivative of Schmidt SNALF can be expressed as

$$\frac{d\bar{P}_n^m(\cos\theta)}{d\theta} = \begin{cases} -\sqrt{\frac{n(n+1)}{2}}\bar{P}_n^1(\cos\theta) & m = 0 \\ -\frac{1}{2}\sqrt{(n-m)(n+m+1)}\bar{P}_n^{m+1}(\cos\theta) & \\ +\frac{1}{2}\sqrt{(1+\delta_0^{m-1})(n+m)(n-m+1)}\bar{P}_n^{m-1}(\cos\theta) & m \geq 1 \end{cases} \tag{31}$$

3.2 Development of the non-singular second-order derivative of Schmidt SNALF

Seeking the derivation of Equation with respect to θ, we can obtain

$$\frac{d^2P_n^m(\cos\theta)}{d\theta^2} = -\frac{1}{2}\frac{dP_n^{m+1}(\cos\theta)}{d\theta} + \frac{1}{2}(n+m)(n-m+1)\frac{dP_n^{m-1}(\cos\theta)}{d\theta} \tag{32}$$

According to the Equation , we have

$$\frac{dP_n^{m+1}(\cos\theta)}{d\theta} = -\frac{1}{2}P_n^{m+2}(\cos\theta) + \frac{1}{2}(n-m)(n+m+1)P_n^m(\cos\theta) \tag{33}$$

$$\frac{dP_n^{m-1}(\cos\theta)}{d\theta} = -\frac{1}{2}P_n^m(\cos\theta) + \frac{1}{2}(n-m+2)(n+m-1)P_n^{m-2}(\cos\theta) \tag{34}$$

Substituting Equations and into Equation , the non-singular formula of the second-order derivative of Legendre polynomial when $m>1$ can be obtained as follows

$$\begin{aligned}\frac{d^2P_n^m(\cos\theta)}{d\theta^2} &= \frac{1}{4}P_n^{m+2}(\cos\theta)-\frac{1}{2}(n^2+n-m^2)P_n^m(\cos\theta)+ \\ &\quad \frac{1}{4}(n+m-1)(n+m)(n-m+1)(n-m+2)P_n^{m-2}(\cos\theta)\end{aligned} \tag{35}$$

Seeking the derivation of Equation with respect to θ, we can obtain

$$\frac{d^2P_n^m(\cos\theta)}{d\theta^2} = -\frac{dP_n^{m+1}(\cos\theta)}{d\theta}+\frac{dP_n^m(\cos\theta)}{d\theta}\frac{m\cos\theta}{\sin\theta}-\frac{m}{\sin^2\theta}P_n^m(\cos\theta) \tag{36}$$

According to Equation , we can write

$$\frac{dP_n^{m+1}(\cos\theta)}{d\theta} = -P_n^{m+2}(\cos\theta)+\frac{(m+1)\cos\theta}{\sin\theta}P_n^{m+1}(\cos\theta) \tag{37}$$

Substituting Equations and into Equation , we can derive

$$\frac{d^2P_n^m(\cos\theta)}{d\theta^2} = P_n^{m+2}(\cos\theta)-\frac{(2m+1)\cos\theta}{\sin\theta}P_n^{m+1}(\cos\theta)+\frac{m(m\cos^2\theta-1)}{\sin^2\theta}P_n^m(\cos\theta) \tag{38}$$

Therefore, in Equation , when m=0, there is

$$\frac{d^2P_n^0(\cos\theta)}{d\theta^2} = P_n^2(\cos\theta)-\frac{\cos\theta}{\sin\theta}P_n^1(\cos\theta) \tag{39}$$

According to Equation , we can obtain

$$\frac{\cos\theta}{\sin\theta}P_n^1(\cos\theta) = \frac{1}{2}P_n^2(\cos\theta)+\frac{1}{2}n(n+1)P_n^0(\cos\theta) \tag{40}$$

Substituting Equation into Equation , the non-singular formula of the second-order derivative of Legendre polynomial when m=0 can be obtained as follows

$$\frac{d^2P_n^0(\cos\theta)}{d\theta^2} = \frac{1}{2}P_n^2(\cos\theta)-\frac{1}{2}n(n+1)P_n^0(\cos\theta) \tag{41}$$

In Equation , when m=1, there is

$$\frac{d^2P_n^1(\cos\theta)}{d\theta^2} = P_n^3(\cos\theta)-\frac{3\cos\theta}{\sin\theta}P_n^2(\cos\theta)-P_n^1(\cos\theta) \tag{42}$$

According to Equation , there is

$$\frac{2\cos\theta}{\sin\theta}P_n^2(\cos\theta) = \frac{1}{2}P_n^3(\cos\theta)+\frac{1}{2}(n-1)(n+2)P_n^1(\cos\theta) \tag{43}$$

Substituting Equation into Equation , the non-singular formula of the second-order derivative of Legendre polynomial when m=1 can be obtained as follows

$$\frac{d^2P_n^1(\cos\theta)}{d\theta^2} = \frac{1}{4}P_n^3(\cos\theta)-\frac{1}{4}(3n^2+3n-2)P_n^1(\cos\theta) \tag{44}$$

Similarly, according to the Equation , the Legendre polynomials in Equations , , and are Schmidt semi-standardized and the non-singular formulae of the second-order derivative of Schmidt SNALF can be expressed as

$$\frac{d^2\overline{P}_n^m(\cos\theta)}{d\theta^2}=\begin{cases}\frac{1}{2}\sqrt{\frac{n(n-1)(n+1)(n+2)}{2}}\overline{P}_n^2(\cos\theta)-\frac{1}{2}n(n+1)\overline{P}_n^0(\cos\theta) & m=0\\ \frac{1}{4}\sqrt{(n-2)(n-1)(n+2)(n+3)}\overline{P}_n^3(\cos\theta) & \\ -\frac{1}{4}(3n^2+3n-2)\overline{P}_n^1(\cos\theta) & m=1\\ \frac{1}{4}\sqrt{(n-m-1)(n-m)(n+m+1)(n+m+2)}\overline{P}_n^{m+2}(\cos\theta)+ & \\ \frac{1}{4}\sqrt{(1+\delta_0^{m-2})(n+m-1)(n+m)(n-m+1)(n-m+2)}\overline{P}_n^{m-2}(\cos\theta)- & \\ \frac{1}{2}(n^2+n-m^2)\overline{P}_n^m(\cos\theta) & m\geq 2\end{cases}\tag{45}$$

3.3 Development of the non-singular first kind spherical harmonic polynomial

According to the differential equation of Legendre polynomial (Heiskanen and Moritz, 1967; Moritz, 1980; Bernhard and Moritz, 2006), there is

$$\sin\theta\frac{d^2P_n^m(\cos\theta)}{d\theta^2}+\cos\theta\frac{dP_n^m(\cos\theta)}{d\theta}+\left[n(n+1)\sin\theta-\frac{m^2}{\sin\theta}\right]P_n^m(\cos\theta)=0\tag{46}$$

When $m\geq1$, we can obtain

$$\frac{mP_n^m(\cos\theta)}{\sin\theta}=\frac{1}{m}\left[n(n+1)\sin\theta P_n^m(\cos\theta)+\cos\theta\frac{dP_n^m(\cos\theta)}{d\theta}+\sin\theta\frac{d^2P_n^m(\cos\theta)}{d\theta^2}\right]\tag{47}$$

Substituting Equations and into Equation , we can derive

$$\begin{aligned}\frac{mP_n^m(\cos\theta)}{\sin\theta}=&\frac{1}{2m}(n^2+n+m^2)\sin\theta P_n^m(\cos\theta)-\frac{1}{2m}\cos\theta P_n^{m+1}(\cos\theta)+\\&\frac{1}{2m}(n+m)(n-m+1)\cos\theta P_n^{m-1}(\cos\theta)+\frac{1}{4m}\sin\theta P_n^{m+2}(\cos\theta)+\\&\frac{1}{4m}(n+m-1)(n+m)(n-m+1)(n-m+2)\sin\theta P_n^{m-2}(\cos\theta)\end{aligned}\tag{48}$$

According to Equation , we can write

$$P_n^{m-2}(\cos\theta)=\frac{2(m-1)}{(n-m+2)(n+m-1)}\frac{\cos\theta}{\sin\theta}P_n^{m-1}(\cos\theta)-\frac{1}{(n-m+2)(n+m-1)}P_n^m(\cos\theta)\tag{49}$$

Substituting Equations and into Equation , we can obtain

$$\frac{mP_n^m(\cos\theta)}{\sin\theta}=\frac{1}{2}\cos\theta\left[P_n^{m+1}(\cos\theta)+(n+m)(n-m+1)P_n^{m-1}(\cos\theta)\right]+m\sin\theta P_n^m(\cos\theta)\tag{50}$$

The Legendre polynomial in Equation is Schmidt semi-standardized and the non-singular formula of the first kind spherical harmonic polynomial can be expressed as

$$\begin{aligned}\frac{m\bar{P}_n^m(\cos\theta)}{\sin\theta} &= \frac{1}{2}\cos\theta[\sqrt{(n-m)(n+m+1)}\bar{P}_n^{m+1}(\cos\theta)+ \\ &\sqrt{(1+\delta_0^{m-1})(n+m)(n-m+1)}\bar{P}_n^{m-1}(\cos\theta)] + m\sin\theta\bar{P}_n^m(\cos\theta)\end{aligned} \tag{51}$$

3.4 Development of the non-singular second kind spherical harmonic polynomial

According to Equation , when $m \neq 0$, we can obtain

$$\frac{m^2 P_n^m(\cos\theta)}{\sin^2\theta} = n(n+1)P_n^m(\cos\theta) + \frac{\cos\theta}{\sin\theta}\frac{dP_n^m(\cos\theta)}{d\theta} + \frac{d^2 P_n^m(\cos\theta)}{d\theta^2} \tag{52}$$

Substituting Equations and into Equation , we can obtain

$$\begin{aligned}\frac{m^2 P_n^m(\cos\theta)}{\sin^2\theta} &= \frac{1}{2}(n^2+n+m^2)P_n^m(\cos\theta) - \frac{\cos\theta}{2\sin\theta}P_n^{m+1}(\cos\theta) + \\ &\frac{1}{2}(n+m)(n-m+1)\frac{\cos\theta}{\sin\theta}P_n^{m-1}(\cos\theta) + \frac{1}{4}P_n^{m+2}(\cos\theta) + \\ &\frac{1}{4}(n+m-1)(n+m)(n-m+1)(n-m+2)P_n^{m-2}(\cos\theta)\end{aligned} \tag{53}$$

According to Equation , when $m \geqslant 2$, we can obtain

$$\begin{aligned}\frac{(m+1)P_n^{m+1}(\cos\theta)}{\sin\theta} &= \frac{1}{2}\cos\theta\left[P_n^{m+2}(\cos\theta) + (n+m+1)(n-m)P_n^m(\cos\theta)\right] + \\ &(m+1)\sin\theta P_n^{m+1}(\cos\theta)\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{(m-1)P_n^{m-1}(\cos\theta)}{\sin\theta} &= \frac{1}{2}\cos\theta\left[P_n^m(\cos\theta) + (n+m-1)(n-m+2)P_n^{m-2}(\cos\theta)\right] + \\ &(m-1)\sin\theta P_n^{m-1}(\cos\theta)\end{aligned} \tag{55}$$

Substituting Equations and into Equation , we can obtain

$$\begin{aligned}\frac{m^2 P_n^m(\cos\theta)}{\sin^2\theta} &= \left[\frac{1}{2}(n^2+n+m^2) + \frac{n(n+1)}{2(m^2-1)}\cos^2\theta\right]P_n^m(\cos\theta) - \\ &\frac{1}{2}\sin\theta\cos\theta P_n^{m+1}(\cos\theta) + \frac{1}{4}\left[1 - \frac{\cos^2\theta}{m+1}\right]P_n^{m+2}(\cos\theta) + \\ &\frac{1}{2}(n+m)(n-m+1)\sin\theta\cos\theta P_n^{m-1}(\cos\theta) + \\ &\frac{1}{4}(n+m-1)(n+m)(n-m+1)(n-m+2)\left[1 + \frac{\cos^2\theta}{m-1}\right]P_n^{m-2}(\cos\theta)\end{aligned} \tag{56}$$

The Legendre polynomial in Equation is Schmidt semi-standardized and the non-singular formula of the second kind spherical harmonic polynomial can be expressed as

$$
\begin{aligned}
\frac{m^2\bar{P}_n^m(\cos\theta)}{\sin^2\theta} = & \frac{1}{4}\left[1+\frac{\cos^2\theta}{m-1}\right]\sqrt{(1+\delta_0^{m-2})(n+m-1)(n+m)(n-m+1)(n-m+2)}\bar{P}_n^{m-2}(\cos\theta)+ \\
& \frac{1}{2}\sqrt{(n+m)(n-m+1)}\sin\theta\cos\theta\bar{P}_n^{m-1}(\cos\theta)+ \\
& \frac{1}{2}\left[(n^2+n+m^2)+\frac{n(n+1)}{m^2-1}\cos^2\theta\right]\bar{P}_n^m(\cos\theta)- \\
& \frac{1}{2}\sqrt{(n+m+1)(n-m)}\sin\theta\cos\theta\bar{P}_n^{m+1}(\cos\theta)+ \\
& \frac{1}{4}\left[1-\frac{\cos^2\theta}{m+1}\right]\sqrt{(n-m-1)(n-m)(n+m+1)(n+m+2)}\bar{P}_n^{m+2}(\cos\theta)
\end{aligned}
\tag{57}
$$

3.5 New non-singular expressions of GVs

From Equation we can see that there is no singularity in B_z, and we only have to deduce the non-singular formulae of B_x and B_y.

Substituting Equation into Equation , the explicit non-singular formulae of B_x can be expressed as

$$
B_x = \sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m\cos m\lambda + h_n^m\sin m\lambda\right]\cdot
\begin{cases}
-\sqrt{\frac{n(n+1)}{2}}\bar{P}_n^1(\cos\theta) & m=0 \\
-\frac{1}{2}\sqrt{(n-1)(n+2)}\bar{P}_n^2(\cos\theta)+\frac{1}{2}\sqrt{2(n+1)}\bar{P}_n^0(\cos\theta) & m=1 \\
-\frac{1}{2}\sqrt{(n-m)(n+m+1)}\bar{P}_n^{m+1}(\cos\theta)+\frac{1}{2}\sqrt{(n+m)(n-m+1)}\bar{P}_n^{m-1}(\cos\theta) & n>m>1 \\
\frac{1}{2}\sqrt{2n}\bar{P}_n^{n-1}(\cos\theta) & m=n>1
\end{cases}
\tag{58}
$$

Substituting Equation into Equation , the explicit non-singular formulae of B_y can be expressed as

$$B_y = \sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+2}\sum_{m=0}^{n}\left[g_n^m \sin m\lambda - h_n^m \cos m\lambda\right]\cdot$$

$$\begin{cases} 0 & m=0 \\ \frac{1}{2}\cos\theta[\sqrt{(n-1)(n+2)}\bar{P}_n^2(\cos\theta)+\sqrt{2n(n+1)}\bar{P}_n^0(\cos\theta)]+\sin\theta\bar{P}_n^1(\cos\theta) & m=1 \\ \frac{1}{2}\cos\theta[\sqrt{(n-m)(n+m+1)}\bar{P}_n^{m+1}(\cos\theta)+ & \\ \sqrt{(n+m)(n-m+1)}\bar{P}_n^{m-1}(\cos\theta)]+m\sin\theta\bar{P}_n^m(\cos\theta) & n>m>1 \\ \frac{\sqrt{2n}}{2}\cos\theta\bar{P}_n^{n-1}(\cos\theta)+n\sin\theta\bar{P}_n^n(\cos\theta) & m=n>1 \end{cases} \tag{59}$$

3.6 New non-singular expressions of GGTs

It can be seen from Equation that there is no singularity in B_{zz}, we only need to seek the non-singular formulae of the other five components of GGTs. Below, using the equations above, we give non-singular formulae of these five components one by one in details.

Substituting Equation into Equation , the explicit non-singular formulae of B_{xx} can be expressed as

$$B_{xx} = \frac{1}{R}\sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \cos m\lambda + h_n^m \sin m\lambda\right]\cdot$$

$$\begin{cases} \frac{1}{2}\sqrt{\frac{n(n-1)(n+1)(n+2)}{2}}\bar{P}_n^2(\cos\theta)-\frac{1}{2}(n+1)(n+2)\bar{P}_n^0(\cos\theta) & m=0 \\ \frac{1}{4}\sqrt{(n-2)(n-1)(n+2)(n+3)}\bar{P}_n^3(\cos\theta)-\frac{1}{4}(3n+1)(n+2)\bar{P}_n^1(\cos\theta) & m=1 \\ \frac{1}{4}\sqrt{(n-3)(n-2)(n+3)(n+4)}\bar{P}_n^4(\cos\theta)+ & \\ \frac{1}{4}\sqrt{2n(n+1)(n+2)(n-1)}\bar{P}_n^0(\cos\theta)-\frac{1}{2}(n^2+3n-2)\bar{P}_n^2(\cos\theta) & m=2 \\ \frac{1}{4}\sqrt{(n-m-1)(n-m)(n+m+1)(n+m+2)}\bar{P}_n^{m+2}(\cos\theta)+ & \\ \frac{1}{4}\sqrt{(n+m-1)(n+m)(n-m+1)(n-m+2)}\bar{P}_n^{m-2}(\cos\theta)- & \\ \frac{1}{2}(n^2+3n-m^2+2)\bar{P}_n^m(\cos\theta) & n>m>2 \\ \frac{1}{2}\sqrt{n(2n-1)}\bar{P}_n^{n-2}(\cos\theta)-\frac{1}{2}(3n+2)\bar{P}_n^n(\cos\theta) & m=n>2 \end{cases} \tag{60}$$

Substituting Equations and into Equation , the explicit non-singular formulae of B_{yy} can be expressed as

$$B_{yy}=\frac{1}{R}\sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m\cos m\lambda+h_n^m\sin m\lambda\right]\cdot$$

$$\begin{cases} -\frac{1}{2}\sqrt{\frac{n(n-1)(n+1)(n+2)}{2}}\cos^2\theta\bar{P}_n^2(\cos\theta)- \\ \sqrt{\frac{n(n+1)}{2}}\sin\theta\cos\theta\bar{P}_n^1(\cos\theta)- \\ \frac{1}{2}(n+1)(n\cos^2\theta+2)\bar{P}_n^0(\cos\theta) & m=0 \\ \\ -\frac{1}{4}\sqrt{(n-2)(n-1)(n+2)(n+3)}\cos^2\theta\bar{P}_n^3(\cos\theta)- \\ \sqrt{(n-1)(n+2)}\sin\theta\cos\theta\bar{P}_n^2(\cos\theta)- \\ \frac{1}{4}\left[(n-1)\cos^2\theta+4\right](n+2)\bar{P}_n^1(\cos\theta) & m=1 \\ \\ -\frac{1}{4}\sqrt{(n-3)(n-2)(n+3)(n+4)}\bar{P}_n^4(\cos\theta)- \\ \frac{1}{2}(n^2+3n+6)\bar{P}_n^2(\cos\theta)- \\ \frac{1}{4}\sqrt{2n(n-1)(n+1)(n+2)}\bar{P}_n^0(\cos\theta) & m=2 \\ \\ -\frac{1}{4}\sqrt{(n-m-1)(n-m)(n+m+1)(n+m+2)}\bar{P}_n^{m+2}(\cos\theta)- \\ \frac{1}{4}\sqrt{(n+m-1)(n+m)(n-m+1)(n-m+2)}\bar{P}_n^{m-2}(\cos\theta)- \\ \frac{1}{2}(n^2+3n+m^2+2)\bar{P}_n^m(\cos\theta) & n>m>2 \\ \\ -\frac{1}{2}\sqrt{n(2n-1)}\bar{P}_n^{n-2}(\cos\theta)-\frac{1}{2}(2n^2+3n+2)\bar{P}_n^n(\cos\theta) & m=n>2 \end{cases} \tag{61}$$

Substituting Equations and into Equation , the explicit non-singular formulae of B_{xy} can be expressed as

$$B_{xy} = -\frac{1}{R}\sum_{n=1}^{N}\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m \sin m\lambda - h_n^m \cos m\lambda\right]\cdot$$

$$\begin{cases}
0 & m=0 \\[2ex]
-\frac{1}{2}\sqrt{2n(n+1)}\sin\theta \overline{P}_n^0(\cos\theta) + \\
\frac{1}{8}\left[(1+\cos^2\theta)(n-1)(n+2)+8\right]\cos\theta \overline{P}_n^1(\cos\theta) + \\
\frac{1}{2}(1+\cos^2\theta)\sin\theta\sqrt{(n-1)(n+2)}\overline{P}_n^2(\cos\theta) + \\
\frac{1}{8}(1+\cos^2\theta)\cos\theta\sqrt{(n-2)(n-1)(n+2)(n+3)}\overline{P}_n^3(\cos\theta) & m=1 \\[2ex]
\frac{1}{8}\cos\theta\sqrt{2n(n-1)(n+1)(n+2)}\left[\cos^2\theta-3\right]\overline{P}_n^0(\cos\theta) + \\
\frac{1}{4}\sin\theta\sqrt{(n-1)(n+2)}\left[\cos^2\theta-4\right]\overline{P}_n^1(\cos\theta) + \\
\frac{1}{12}\cos\theta\left[(12-n^2-n)+n(n+1)\cos^2\theta\right]\overline{P}_n^2(\cos\theta) + \\
\frac{1}{4}\sin\theta\sqrt{(n-2)(n+3)}\left[4-\cos^2\theta\right]\overline{P}_n^3(\cos\theta) + \\
\frac{1}{24}\cos\theta\sqrt{(n-3)(n-2)(n+3)(n+4)}\left[7-\cos^2\theta\right]\overline{P}_n^4(\cos\theta) & m=2 \\[2ex]
\frac{1}{2m}\cos\theta\left[(n^2+n+m^2)-\frac{m^2(n^2+n)}{m^2-1}+\frac{(n^2+n)}{m^2-1}\cos^2\theta\right]\overline{P}_n^m(\cos\theta) + \\
\frac{1}{2m}\sin\theta\sqrt{(n-m)(n+m+1)}\left[m^2-\cos^2\theta\right]\overline{P}_n^{m+1}(\cos\theta) + \\
\frac{1}{4}\cos\theta\sqrt{(n-m-1)(n-m)(n+m+1)(n+m+2)}\cdot \\
\left[\frac{m^2+m+1-\cos^2\theta}{m(m+1)}\right]\overline{P}_n^{m+2}(\cos\theta) + \\
\frac{1}{4}\cos\theta\sqrt{(n+m-1)(n+m)(n-m+1)(n-m+2)}\cdot \\
\left[\frac{\cos^2\theta+m-m^2-1}{m(m-1)}\right]\overline{P}_n^{m-2}(\cos\theta) + \\
\frac{1}{2m}\sin\theta\sqrt{(n+m)(n-m+1)}\left[\cos^2\theta-m^2\right]\overline{P}_n^{m-1}(\cos\theta) & n>m>2 \\[2ex]
\frac{1}{2}\cos\theta\left[n-\frac{\sin^2\theta}{n-1}\right]\overline{P}_n^n(\cos\theta)+\frac{\sqrt{2n}}{2n}\sin\theta\left[\cos^2\theta-n^2\right]\overline{P}_n^{n-1}(\cos\theta) - \\
\frac{1}{2}\cos\theta\sqrt{n(2n-1)}\left[1+\frac{\sin^2\theta}{n(n-1)}\right]\overline{P}_n^{n-2}(\cos\theta) & m=n
\end{cases} \tag{62}$$

Substituting Equation into Equation , the explicit non-singular formulae of B_{xz} can be expressed as

$$B_{xz} = -\frac{1}{R}\sum_{n=1}^{N}(n+2)\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m\cos m\lambda + h_n^m \sin m\lambda\right]\cdot \begin{cases} -\sqrt{\dfrac{n(n+1)}{2}}\bar{P}_n^1(\cos\theta) & m=0 \\ \sqrt{\dfrac{n(n+1)}{2}}\bar{P}_n^0(\cos\theta) - \dfrac{1}{2}\sqrt{(n-1)(n+2)}\bar{P}_n^2(\cos\theta) & m=1 \\ \dfrac{1}{2}\sqrt{(n+m)(n-m+1)}\bar{P}_n^{m-1}(\cos\theta) - \dfrac{1}{2}\sqrt{(n-m)(n+m+1)}\bar{P}_n^{m+1}(\cos\theta) & n>m>1 \\ \dfrac{1}{2}\sqrt{2n}\bar{P}_n^{n-1}(\cos\theta) & m=n>1 \end{cases} \tag{63}$$

Substituting Equation into Equation , the explicit non-singular formulae of B_{yz} can be expressed as

$$B_{yz} = -\frac{1}{R}\sum_{n=1}^{N}(n+2)\left(\frac{R}{r}\right)^{n+3}\sum_{m=0}^{n}\left[g_n^m\sin m\lambda - h_n^m \cos m\lambda\right]\cdot \begin{cases} 0 & m=0 \\ \dfrac{1}{2}\cos\theta[\sqrt{(n-1)(n+2)}\bar{P}_n^2(\cos\theta) + \sqrt{2n(n+1)}\bar{P}_n^0(\cos\theta)] + \\ \sin\theta\bar{P}_n^1(\cos\theta) & m=1 \\ \dfrac{1}{2}\cos\theta[\sqrt{(n-m)(n+m+1)}\bar{P}_n^{m+1}(\cos\theta) + \\ \sqrt{(n+m)(n-m+1)}\bar{P}_n^{m-1}(\cos\theta)] + m\sin\theta\bar{P}_n^m(\cos\theta) & n>m>1 \\ \dfrac{\sqrt{2n}}{2}\cos\theta\bar{P}_n^{n-1}(\cos\theta) + n\sin\theta\bar{P}_n^n(\cos\theta) & m=n>1 \end{cases} \tag{64}$$

4 New expressions of the Recursive calculation of Schmidt SNALF

At present, the commonly used recursive formulae of Legendre polynomials include SFRR, SFCR, CDOR, and BR, which are Schmidt semi-normalized in this section. Then these recursive formulae of Schmidt SNALF can be used to improve the speed and stability in calculating GVs and GGTs.

(1) Development of the recursive formulae of the SFRR

According to Equation , Equation is Schmidt semi-standardized and the recursive formulae of the SFRR can be formulated as

$$\bar{P}_n^m(\cos\theta)=\begin{cases}\dfrac{1}{\sqrt{(1+\delta_0^m)(n-m)(n+m+1)}}[\dfrac{2(m+1)\cos\theta}{\sin\theta}\bar{P}_n^{m+1}(\cos\theta)- \\ \sqrt{(n+m+2)(n-m-1)}\bar{P}_n^{m+2}(\cos\theta)] & n>m \\ \\ \sqrt{\dfrac{(2m-1)}{(2-\delta_0^{m-1})m}}\sin\theta\bar{P}_{m-1}^{m-1}(\cos\theta) & n=m(m>0)\end{cases} \tag{65}$$

(2) Development of recursive formulae of the SFCR

The recursive formulae of the Legendre polynomial of the SFCR is expressed as (Colombo, 1981)

$$P_n^m(\cos\theta)=\begin{cases}\dfrac{2n-1}{n-m}\cos\theta P_{n-1}^m(\cos\theta)-\dfrac{n+m-1}{n-m}P_{n-2}^m(\cos\theta) & n>m \\ (2m-1)\sin\theta P_{m-1}^{m-1}(\cos\theta) & n=m(m>0)\end{cases} \tag{66}$$

According to Equation , Equation is Schmidt semi-standardized and the recursive formulae of the SFCR can be formulated as

$$\bar{P}_n^m(\cos\theta)=\begin{cases}\dfrac{1}{\sqrt{(n-m)(n+m)}}[(2n-1)\cos\theta\bar{P}_{n-1}^m(\cos\theta)- \\ \sqrt{(n-m-1)(n+m-1)}\bar{P}_{n-2}^m(\cos\theta)] & n>m \\ \\ \sqrt{\dfrac{2m-1}{(2-\delta_0^{m-1})m}}\sin\theta\bar{P}_{m-1}^{m-1}(\cos\theta) & n=m(m>0)\end{cases} \tag{67}$$

(3) Development of recursive formulae of the CDOR

When order m equals to 0 and 1, the Legendre polynomial of the CDOR is calculated by Equation . When $m\geq 2$, the Legendre polynomial of the CDOR can be calculated based on the recursive formulae of the spherical harmonic functions as follows

$$\begin{aligned}P_n^m(\cos\theta)=P_{n-2}^m(\cos\theta)+(n+m-2)(n+m-3)P_{n-2}^{m-2}(\cos\theta)- \\ (n-m+1)(n-m+2)P_n^{m-2}(\cos\theta)\end{aligned} \tag{68}$$

According to Equation , Equation is Schmidt semi-standardized and the recursive formula of the CDOR can be formulated as

$$\begin{aligned}\bar{P}_n^m(\cos\theta)=\frac{1}{\sqrt{(n+m)(n+m-1)}}[\frac{1}{\sqrt{(n-m+1)(n-m+2)}}\bar{P}_{n-2}^m(\cos\theta)+ \\ \sqrt{(1+\delta_0^{m-2})}\sqrt{(n+m-2)(n+m-3)}\bar{P}_{n-2}^{m-2}(\cos\theta)- \\ \sqrt{(1+\delta_0^{m-2})}\sqrt{(n-m+1)(n-m+2)}\bar{P}_n^{m-2}(\cos\theta)]\end{aligned} \tag{69}$$

The values of Schmidt SNALF can be obtained by their linear combination when $m \geq 2$. Moreover, the coefficients in front of $\bar{P}_{n-2}^{m}(\cos\theta)$, $\bar{P}_{n-2}^{m-2}(\cos\theta)$, and $\bar{P}_{n}^{m-2}(\cos\theta)$ are less than 1, so the recursive formula of the CDOR is stable and reliable.

(4) Development of recursive formulae of the BR

In the first three recursive formulae, because their recursive coefficients are a function of degree n and order m, the recursion workload increases dramatically with the increase of n and m. To avoid this, a new abnormal spherical harmonic function is introduced (Belikov et al., 1991, 1992)

$$\hat{P}_{n}^{m}(\cos\theta) = \frac{2^{m} n!}{(n+m)!} P_{n}^{m}\left(\cos\theta\right) \tag{70}$$

and the recursive formulae are given as follows

$$\begin{cases} \hat{P}_{n}^{0}(\cos\theta) = \cos\theta \hat{P}_{n-1}^{0}(\cos\theta) - \dfrac{\sin\theta}{2}\hat{P}_{n-1}^{1}(\cos\theta) & m = 0 \\ \hat{P}_{n}^{m}(\cos\theta) = \cos\theta \hat{P}_{n-1}^{m}(\cos\theta) - \sin\theta\left[\dfrac{1}{4}\hat{P}_{n-1}^{m+1}(\cos\theta) - \hat{P}_{n-1}^{m-1}(\cos\theta)\right] & m > 0 \end{cases} \tag{71}$$

where the recursive initial values are $\hat{P}_{0}^{0}(\cos\theta) = 1$, $\hat{P}_{1}^{0}(\cos\theta) = \cos\theta$, and $\hat{P}_{1}^{1}(\cos\theta) = \sin\theta$. The recursion of $\hat{P}_{n}^{m}(\cos\theta)$ can be realized by using Equation . Obviously, the coefficients of the recursive formulae are independent of degree n and order m, and the absolute value is less than 1, so the recursive formulae are fast and stable.

In practice, it is generally necessary to transform $\hat{P}_{n}^{m}(\cos\theta)$ to $\bar{P}_{n}^{m}(\cos\theta)$. According to Equations and , we can obtain

$$\bar{P}_{n}^{m}\left(\cos\theta\right) = \hat{N}_{n}^{m}\hat{P}_{n}^{m}(\cos\theta) \tag{72}$$

where $\hat{N}_{n}^{m} = \dfrac{\sqrt{\left(2-\delta_{0}^{m}\right)(n+m)!\left(n-m\right)!}}{2^{m} n!}$, and it can be calculated by the following recursive formulae

$$\hat{N}_{n}^{m} = \begin{cases} 1 & m = 0 \\ \sqrt{1-\dfrac{m^{2}}{n^{2}}}\hat{N}_{n-1}^{m} & n > m > 0 \\ \sqrt{1-\dfrac{1}{2n}}\hat{N}_{n-1}^{n-1} & n = m > 1 \end{cases} \tag{73}$$

where the recursive initial values are $\hat{N}_{0}^{0} = \hat{N}_{1}^{0} = \hat{N}_{1}^{1} = 1$.

(5) Checking of the recursive calculation of Schmidt SNALF

According to the characteristics of Schmidt SNALF, for the arbitrary θ we know that

$$f_{n} = \sum_{m=0}^{n}\left[\bar{P}_{n}^{m}\left(\cos\theta\right)\right]^{2} = 1 \qquad \forall\theta \tag{74}$$

Therefore,

$$NA_n = |f_n - 1| \qquad n = 2,3,\cdots,N \tag{75}$$

is taken as the standard to check the calculation accuracy of various recursive formulae. The closer NA_n is to 0, the higher the calculation accuracy of the recursive formulae can achieve. A smaller NA_n represents higher accuracy of numerical calculation.

5 Numerical experiments

5.1 The effectiveness analysis of the new non-singular expressions of GVs and GGTs

For convenience, the traditional formulae (such as the first- and second-order derivatives of Schmidt SNALF and its two kinds of spherical harmonic polynomials, GVs and GGTs) are denoted as M1, the non-singular formulae derived in Du et al. (2015) are denoted as M2, and our newly derived non-singular formulae are denoted as M3 hereinafter.

Since the singularity problem does not exist outside polar regions, the calculation results of the non-singular and traditional formulae should be basically the same. The results by M2, and M3 are compared to those by M1 for case θ=60°, to access their efficientness. The calculation results for case n=2 and 0≤m≤1 are shown in Table 1. When n＞2 and 0≤m≤1, the calculation results of M1, M2, and M3 are similar to those in Table 1, and will not be given again. Similarly, the calculation results of M1, M2, and M3 of two kinds of spherical harmonic polynomials are also the same, and not shown here

Table 1 Comparison of calculation results derived by M1, M2, and M3 when θ=60°

Calculation items	Degree and order	Models	Calculation results
$\frac{d\overline{P}_n^m(\cos\theta)}{d\theta}$	n=2, m=0	M1	-1.299
		M2	-0.650
		M3	-1.299
	n=2, m=1	M1	-0.866
		M2	-0.866
		M3	-0.866
$\frac{d^2\overline{P}_n^m(\cos\theta)}{d\theta^2}$	n=2, m=0	M1	1.500
		M2	0.938
		M3	1.500
	n=2, m=1	M1	-3.000
		M2	-1.875
		M3	-3.000

It is found that, outside polar regions, the results derived by our non-singular formulae are exactly the same as the results by the traditional formulae, proving its validity . However, according to Du et al. (2015), their non-singular formula of the first-order derivative of Schmidt SNALF is incorrect when m=0, and that of the second-order derivative of Schmidt SNALF is incorrect when the order m is equal to 0 and 1. Therefore, the non-singular formulae of B_x and B_{xz} are incorrect when the order m is equal to 0; the non-singular formulae of B_{xx}, B_{yy}, and B_{xy} are incorrect when the order m is equal to 0 and 1.

Based on the geomagnetic field model EMM2017 up to degree 720, the M1, M2, and M3 are used to calculate B_x, B_y, B_{xx}, B_{yy}, B_{xy}, B_{xz}, and B_{yz} for a selected point (60.25°N, 155.75°E) on the surface of the Earth. The results are shown in Tables 2 and 3.

Table 2 Results comparison of GVs calculated by M1, M2, and M3 (Unit: nT)

Calculation items	Models	Calculation results
B_x	M1	14790.63
	M2	8878.44
	M3	14790.63
B_y	M1	-1968.23
	M2	-1968.23
	M3	-1968.23

Table 3 Results comparison of GGTs calculated by M1, M2, and M3 (Unit: nT/km)

Calculation items	Models	Calculation results
B_{xx}	M1	11.648
	M2	11.128
	M3	11.648
B_{yy}	M1	9.111
	M2	9.631
	M3	9.111
B_{xy}	M1	-0.381
	M2	-0.428
	M3	-0.381
B_{xz}	M1	-6.651
	M2	-4.392
	M3	-6.651
B_{yz}	M1	2.164
	M2	2.164
	M3	2.164

As shown in Tables 2 and 3, the calculation results of B_y and B_{yz} by M1, M2, and M3 are the same, for those of the first kind of spherical harmonic polynomial contained in B_y and B_{yz} are the same. The calculation results of B_x, B_{xx}, B_{yy}, B_{xy}, and B_{xz} of M2 are different from those of M1 and M3, because the non-singular formulae from Du et al. (2015) are incorrect for calculating the first- and second-order derivatives of Schmidt SNALF contained in B_x, B_{xx}, B_{yy}, B_{xy}, and B_{xz} when order m is equal to 0 and 1.

5.2 Analysis of the recursive calculation of Schmidt SNALF

(1) Analysis of recursive calculation time

The calculation speed of four recursive formulae of Schmidt SNALF is analyzed. Selecting a special value of θ and calculating it from the degree 1 to the degree 2160 by each recursive formula, the calculation time are presented in Table 4.

Table 4 Time consuming of various recursive formulae (Unit: msec)

Serial number	Recursive formulae	Calculation time
1	SFRR	242
2	SFCR	148
3	CDOR	89
4	BR	481

It can be seen that the calculation speed of the CDOR is the fastest, followed by the SFCR and the SFRR, and the BR is the slowest.

(2) Stability analysis of recursive calculation

We consider both high and low latitude regions, and choose cosθ to be, 0.01, 0.1, 0.9, 0.99. Equation is used to access the calculation stability for each recursive formula. The results are shown in Figs. 1-4.

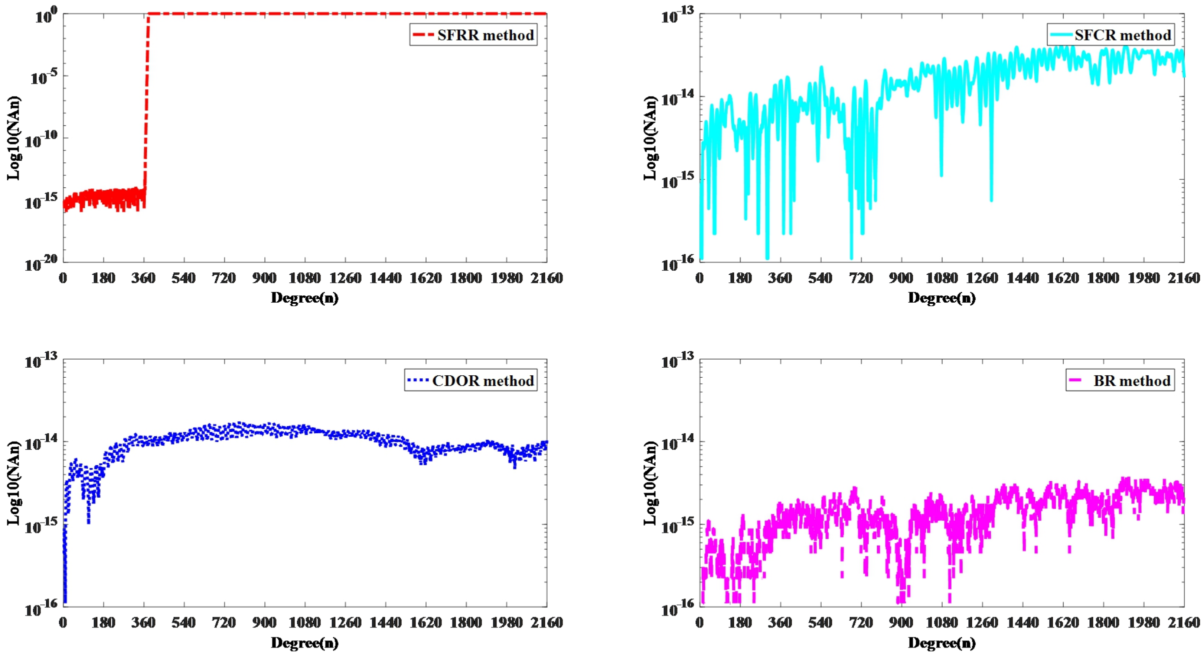

Figure 1. Accuracy checking results of four recursive formulae when cosθ=0.99

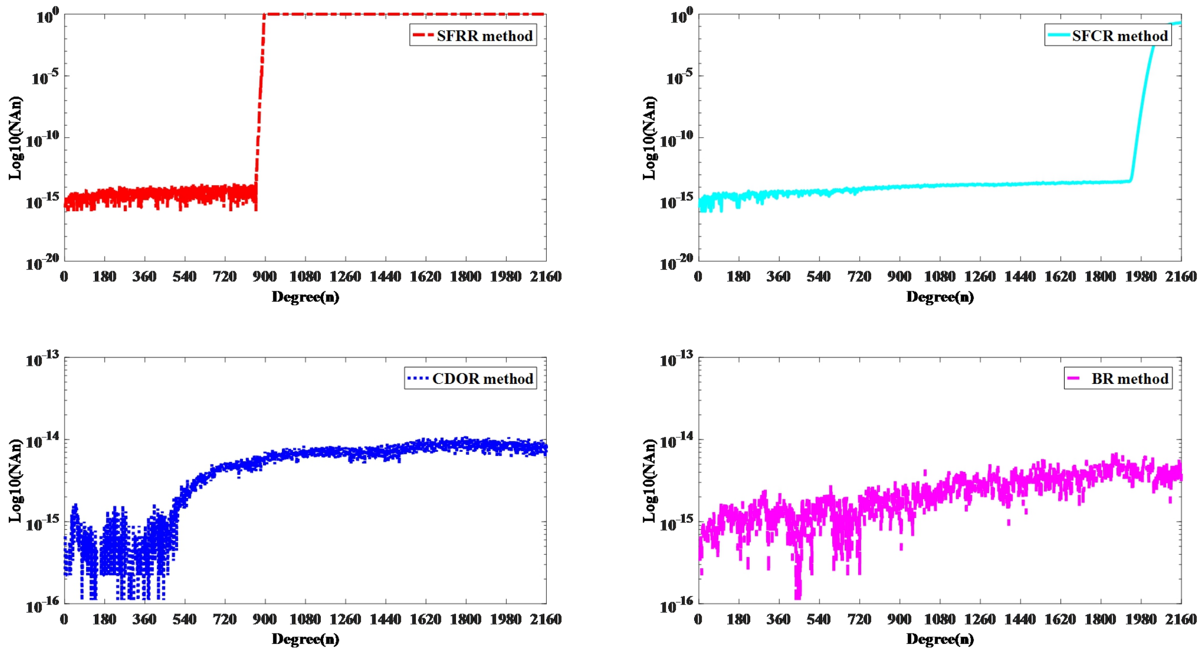

Figure 2. Accuracy checking results of four recursive formulae when cosθ=0.9

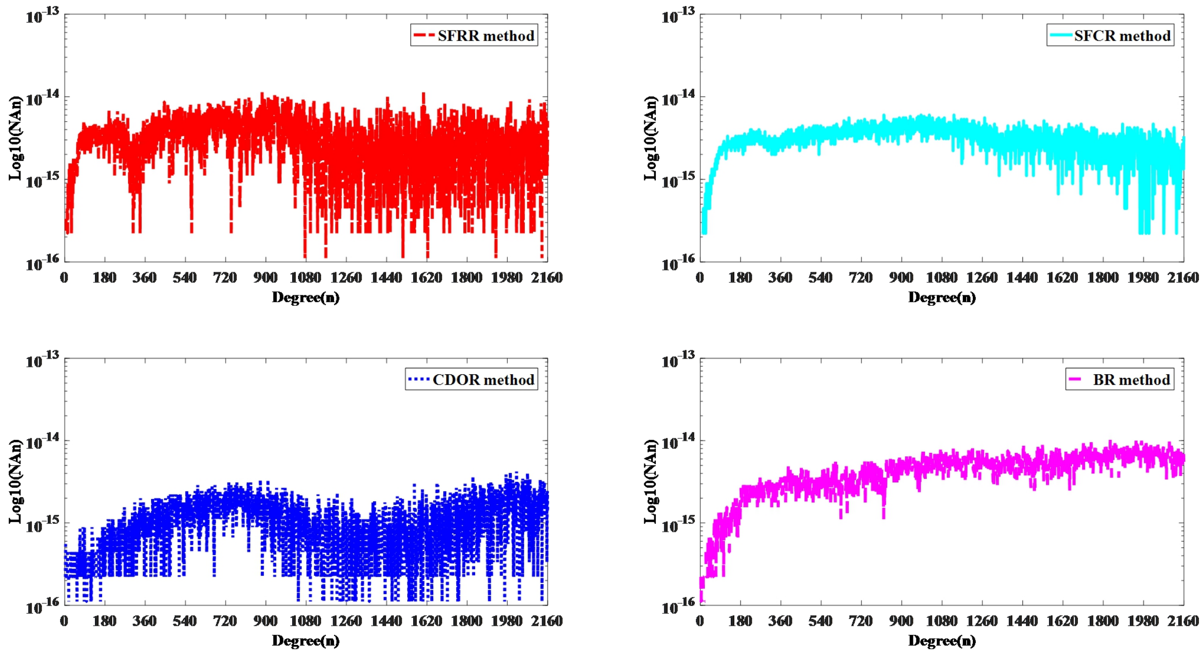

Figure 3. Accuracy checking results of four recursive formulae when cosθ=0.1

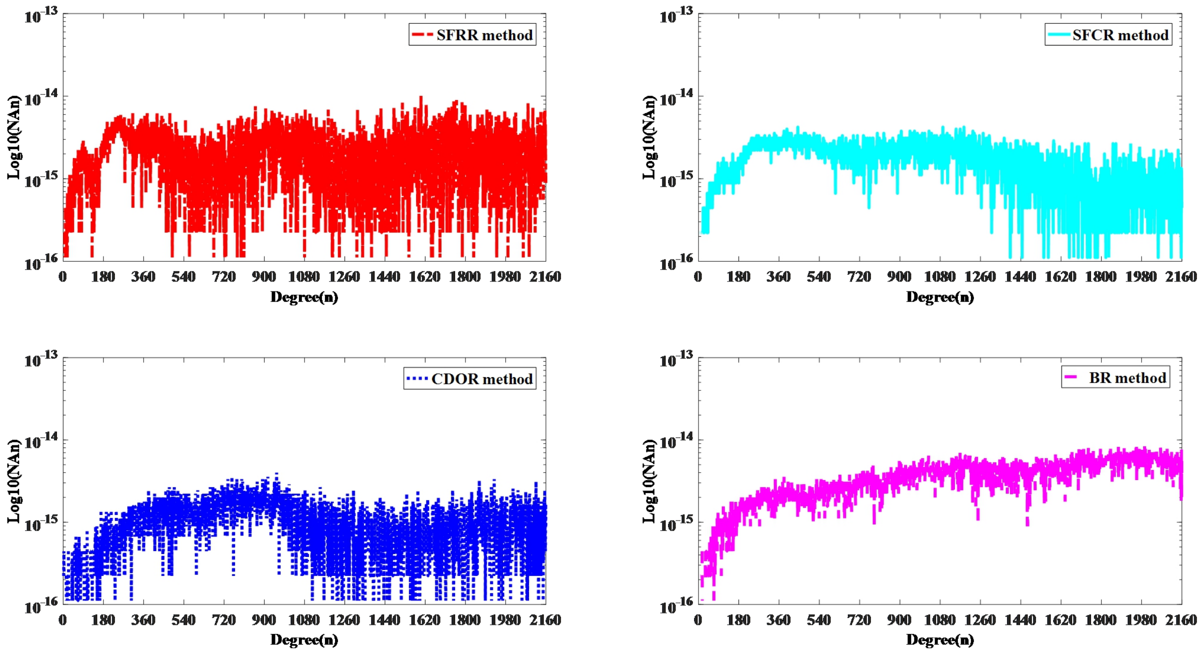

Figure 4. Accuracy checking results of four recursive formulae when cosθ=0.01

It is observed from Figs. 1 and 2 that in high latitude areas, such as cosθ＞0.9, a larger degree n causes extremely unstable recursive calculation of SFRR, because its recursive formula has sinθ in the denominator, which leads to a large calculation error in high latitude areas. Thus, this recursive formula is seldom used in the calculation of Schmidt SNALF.

If the degree n is larger than 1900, the recursive error in the SFCR increases dramatically, as shown in Figure 2. Therefore, this recursive formula can be only used in the calculation of Schmidt SNALF for low degree in low latitude areas.

From Figs. 1 and 2 we find that in high latitude areas, the accuracy of BR is better than that of CDOR. The opposite is true in low latitude areas, as shown in Figs. 3 and 4.

The above four recursive formulae can satisfy the accuracy requirements when calculating Schmidt SNALF for low degree, say n＜360. However, when calculating Schmidt SNALF for high degree in high latitude areas, the accuracy of the SFRR and the SFCR decreases rapidly. On the other hand, the accuracy of the CDOR and the BR is still high, which is not limited by the degree, and can be calculated to degree 2160 or higher.

5.3 Calculation and analysis of GVs and GGTs

(1) Calculation and analysis of GVs

Based on the geomagnetic field model EMM2017 up to degree 720, the GVs terms are calculated for polar regions with orbit height of 300 km, using our newly derived non-singular expressions. The grid resolution is 30′×30′. The statistical results for Arctic region (66.5°N~90°N) and Antarctic region (-90°S~-66.5°S) are shown in Tables 5 and 6, respectively.

Table 5 The statistical results of GVs data in the Arctic region (Unit: nT)

GVs	Maximum	Minimum	Average value	Standard deviation
B_x	13399.07	-1746.16	4376.62	3315.05
B_y	4464.75	-4859.62	1.39×10^{-9}	2034.21
B_z	63419.54	50640.30	56305.76	1728.32

Table 6 The statistical results of GVs data in the Antarctic region (Unit: nT)

GVs	Maximum	Minimum	Average value	Standard deviation
B_x	19501.09	-16536.47	5042.06	10720.05
B_y	16601.79	-17368.02	-8.26×10^{-8}	11357.32
B_z	-30030.44	-66216.28	-50170.52	8246.31

We find that B_z has the largest magnitude in Maximum, Minimum, and average, followed by B_x and B_y. However, B_z has a smaller standard deviations than B_x and B_y. The average value of B_y is substantically smaller than the average value of both B_x and B_y.

The contour map of GVs calculated for Arctic and Antarctic regions are shown in Figs. 5 and 6, respectively. It is observed that the magnitude of each component of GVs is consistent with the analysis in Tables 5 and 6. In addition, according to Figs. 5 and 6, each component of GVs data in the eastern and western hemispheres is symmetrical.

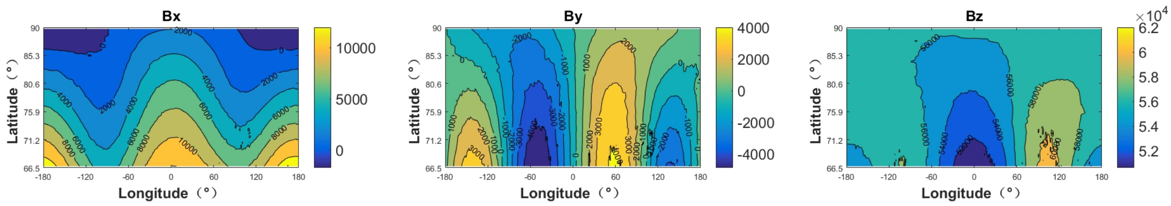

Figure 5. Contour map of each component of GVs data in Arctic region

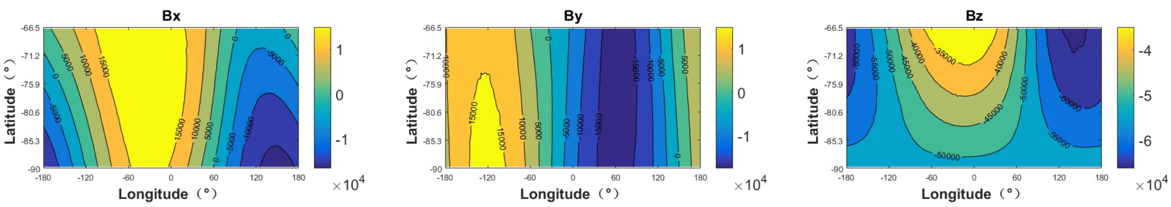

Figure 6. Contour map of each component of GVs data in Antarctic region

(2) Calculation and analysis of GGTs

Theoretically, according to the Laplace equation, $B_{xx}+B_{yy}+B_{zz}=0$. Numerically the closer $B_{xx}+B_{yy}+B_{zz}$ is to 0, the higher accuracy of the non-singular formulae of B_{xx} and B_{yy} can achieve. M1, M2 and M3 are used to calculate the GGTs data for polar regions at the orbit height of 300 km, based on the geomagnetic field model EMM2017 up to degree 720. The regions are gridded into a 30′×30′ grid. The accuracy statistics of $B_{xx}+B_{yy}+B_{zz}$ in Arctic and

Antarctic regions are shown in Tables 7 and 8, respectively.

Table 7 The accuracy statistics of $B_{xx}+B_{yy}+B_{zz}$ in Arctic region (Unit: nT/km)

Models	Maximum	Minimum	Average value	Standard deviation
M1	1.279×10^{-13}	-1.279×10^{-13}	-9.718×10^{-16}	3.332×10^{-14}
M2	1.421×10^{-13}	-1.350×10^{-13}	1.124×10^{-15}	3.348×10^{-14}
M3	1.279×10^{-13}	-1.315×10^{-13}	-9.040×10^{-16}	3.332×10^{-14}

Table 8 The accuracy statistics of $B_{xx}+B_{yy}+B_{zz}$ in Antarctic region (Unit: nT/km)

Models	Maximum	Minimum	Average value	Standard deviation
M1	1.243×10^{-13}	-1.528×10^{-13}	-4.273×10^{-17}	3.078×10^{-14}
M2	1.208×10^{-13}	-1.315×10^{-13}	-2.926×10^{-16}	3.045×10^{-14}
M3	1.208×10^{-13}	-1.315×10^{-13}	2.074×10^{-16}	3.031×10^{-14}

As shown in Tables 7 and 8, we find that the results by M3 numerically satisfies the Laplace equation in polar regions with very high accuracy, and are better than those of M1 and M2.

The statistical results of GGTs data in Arctic and Antarctic regions, which are simulated by our newly derived non-singular formulae, are shown in Tables 9 and 10.

Table 9 The statistical results of GGTs data in the Arctic region (Unit: nT/km)

GGTs	Maximum	Minimum	Average value	Standard deviation
B_{xx}	12.607	8.808	10.646	0.973
B_{yy}	12.794	8.279	10.245	1.113
B_{zz}	-18.310	-25.006	-20.890	1.278
B_{xy}	1.447	-1.367	1.327×10^{-17}	0.819
B_{xz}	1.381	-5.099	-1.056	1.400
B_{yz}	2.599	-2.969	5.463×10^{-17}	1.326

Table 10 The statistical results of GGTs data in the Antarctic region (Unit: nT/km)

GGTs	Maximum	Minimum	Average value	Standard deviation
B_{xx}	-2.568	-14.542	-9.593	2.697
B_{yy}	-4.921	-13.594	-9.717	1.888
B_{zz}	27.847	8.367	19.310	4.476
B_{xy}	1.498	-1.238	2.247×10^{-17}	0.680
B_{xz}	8.551	-9.384	-2.124	5.380
B_{yz}	9.322	-8.573	2.011×10^{-16}	5.839

It can be seen that B_{zz} has the largest magnitude, followed by B_{xx} and B_{yy}, then B_{xz} and B_{yz}, and B_{xy} has the smallest magnitude.

The contour maps of the GGTs data in Arctic and Antarctic regions are shown in Figs. 7 and 8. It can be seen that the magnitude of each component of GGTs is consistent with the analysis in Tables 9 and 10. In addition, in Figs. 7 and 8, each component of GGTs data in the eastern and western hemispheres is symmetrical.

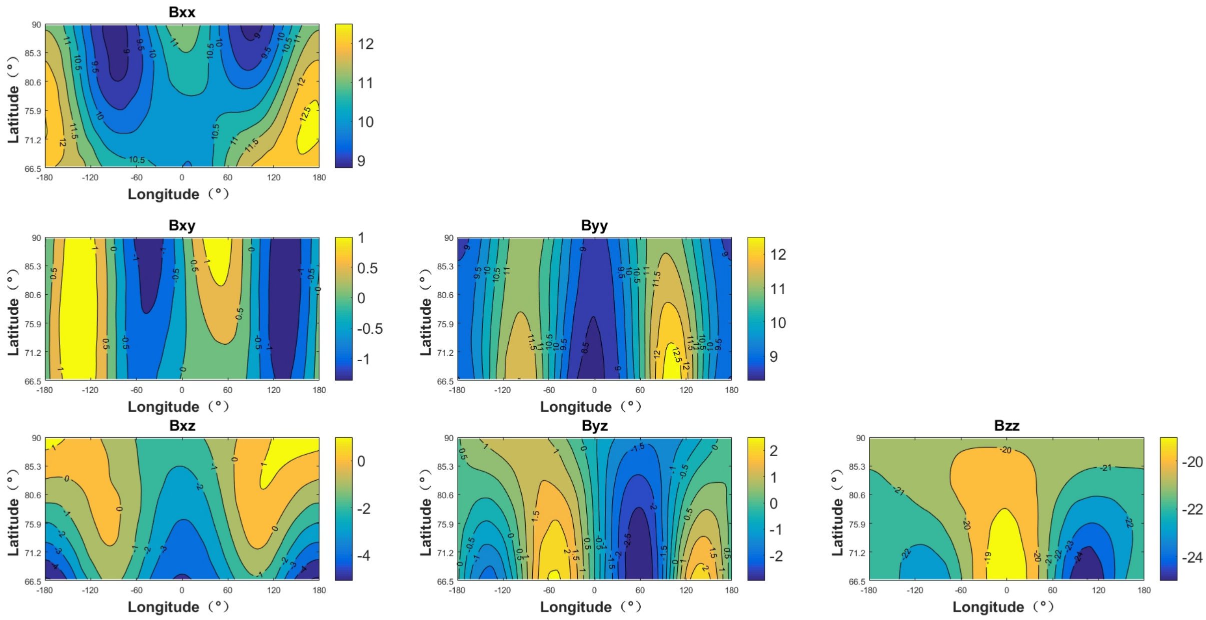

Figure 7. Contour map of GGTs data in Antarctic region

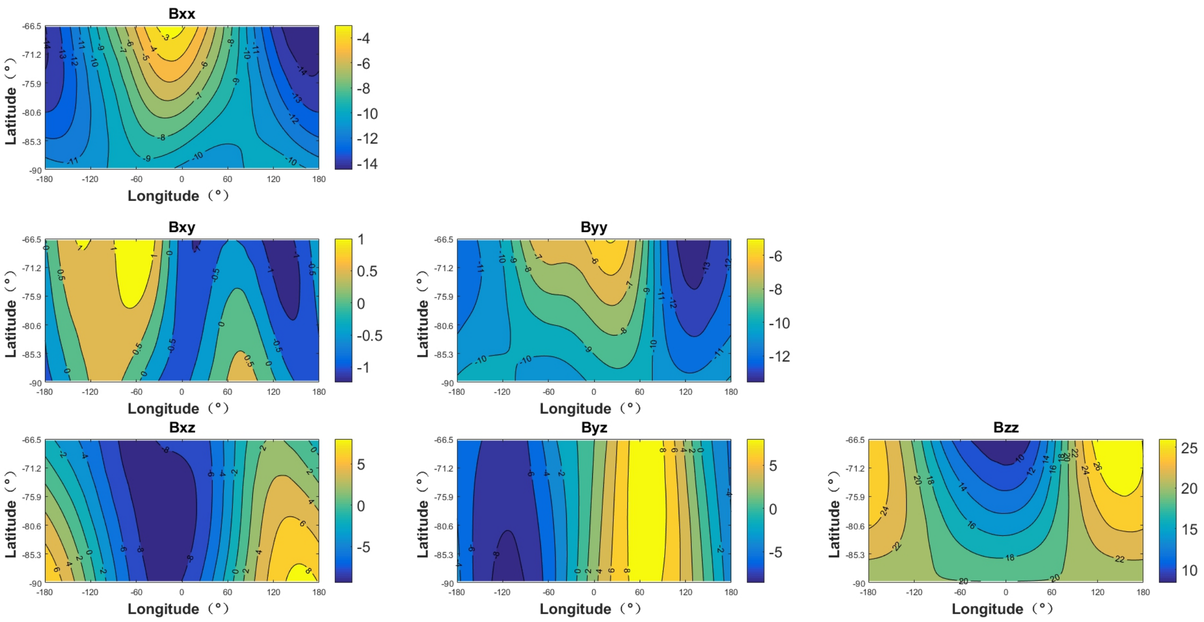

Figure 8. Contour map of GGTs data in Arctic region

6 Conclusion

In this paper, based on a linear combination of Schmidt SNALF, a novel non-singular expressions are derived for the first- and second-order derivatives of Schmidt SNALF, along with its two kinds of spherical harmonic polynomials. When applying above non-singular derivatives and polynomials to traditional formulae of GVs and GGTs, special cases that the order *m* equals to 0, 1, 2 and other values are considered, more practical non-singular expressions of GVs and GGTs are formulated, which achieves significant improvements in solving the singularity problem of the SHEs of GVs and GGTs in polar regions. In addition, four recursive formulae of Schmidt SNALF are derived, and the calculation speed and stability are analyzed and evaluated as well. The application scenarios of these four recursive formulae are also analyzed and the flexible calculation strategies for Schmidt SNALF are presented. The research results can be applied to data processing and modeling of airbore and satellite measurements of GVs and GGTs in polar regions.

Data Availability Statement

The geomagnetic field model EMM2017 can be downloaded from CIRES's website (http://geomag.colorado.edu/).

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Appendix

For the convenience of the formulae derivation, the initial values of the first three orders of Legendre polynomials and Schmidt SNALF are given here, and later orders can be calculated with the initial values of the first three orders.

$$\begin{aligned}
&P_1^0(\cos\theta)=\cos\theta, \qquad P_1^1(\cos\theta)=\sin\theta\\
&P_2^0(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1), \qquad P_2^1(\cos\theta)=3\sin\theta\cos\theta, \qquad P_2^2(\cos\theta)=3\sin^2\theta\\
&P_3^0(\cos\theta)=\frac{1}{2}\cos\theta(5\cos^2\theta-3), P_3^1(\cos\theta)=\frac{3}{2}\sin\theta(5\cos^2\theta-1),\\
&P_3^2(\cos\theta)=15\sin^2\theta\cos\theta, \qquad P_3^3(\cos\theta)=15\sin^3\theta
\end{aligned} \tag{76}$$

$$\begin{aligned}
&\overline{P}_1^0(\cos\theta)=\cos\theta, \qquad \overline{P}_1^1(\cos\theta)=\sin\theta\\
&\overline{P}_2^0(\cos\theta)=\frac{1}{2}(3\cos^2\theta-1), \qquad \overline{P}_2^1(\cos\theta)=\sqrt{3}\sin\theta\cos\theta, \qquad \overline{P}_2^2(\cos\theta)=\frac{\sqrt{3}}{2}\sin^2\theta\\
&\overline{P}_3^0(\cos\theta)=\frac{1}{2}\cos\theta(5\cos^2\theta-3), \overline{P}_3^1(\cos\theta)=\frac{\sqrt{6}}{4}\sin\theta(5\cos^2\theta-1),\\
&\overline{P}_3^2(\cos\theta)=\frac{\sqrt{15}}{2}\sin^2\theta\cos\theta, \qquad \overline{P}_3^3(\cos\theta)=\frac{\sqrt{10}}{4}\sin^3\theta
\end{aligned} \tag{77}$$

References

Balmino, G., Barriot, J., & Valès, N. (1990). Non-singular formulation of the gravity vector and gravity gradient tensor in spherical harmonic, *Manuscript Geodaetica*, *15*, 11-16.

Barraclough, D. R. (1974). Spherical harmonic analyses of the geomagnetic field for eight epochs between 1600 and 1910, *Geophysical Journal of the Royal Astronomical Society*, *36*, 497-513.

Belikov, M. V., & Taybatorov, K. A. (1992). An efficient algorithm for computing the Earth's gravitational potential and its derivatives at satellite altitudes, *Mamuscripta Geodaetica*, *17*, 104-116.

Belikov, M. V. (1991). Spherical harmonic analysis and synthesis with the use of column-wise recurrence relations, *Mamuscripta Geodaetica*, *16*, 384-410.

Benton, E. R., Estes, R. H., Langel, R. A., & Muth, L. A. (1982). Sensitivity of selected geomagnetic properties to truncation level of spherical harmonic expansions, *Geophysical Research Letters*, *9*,254-257.

Bernhard, H. W., & Moritz, H. (2006). *Physical geodesy*, SpringerWienNewYork, Berlin.

Bettadpur, S. V. (1995). Hotine's geopotential formulation: revisited, *Bulletin Géodésique*, *69*, 135-142.

Blakely, R. G. (1995). *Potential theory in gravity and magnetic applications*, Cambridge University Press, New York.

Casotto, S., & Fantino, E. (2007). Evaluation of methods for spherical harmonic synthesis of the gravitational potential and its gradients, *Advances in Space Research*, *40*, 69-75.

Chambodut, A., Pante, I., Mandea, M., Diament, M., Holschneider, M., & Jamet, O. (2005). Wavelet frames: an alternative to spherical harmonic representation of potential fields, *Geophysical Journal International*, *163*, 875-899.

Chen, J. L., Wilson, C. R., & Seo, K.-W. (2006). Optimized smoothing of Gravity Recovery and Climate Experiment (GRACE) time-variable gravity observations, *Journal of Geophysical Research*, *111*, B06408, doi:10.1029/2005JB004064.

Colombo, O. L. (1981). *Numerical methods for harmonic analysis on the sphere*, Department of Geodetic Science and Surveying, the Ohio State University, Columbus.

Du, J. S., Chen, C., Lesur, V., & Wang, L. (2015). Non-singular spherical harmonic expressions of geomagnetic vector and gradient tensor fields in the local north-oriented reference frame, *Geoscientific Model Development*, *8*, 1979-1990.

Eshagh, M. (2008). Non-singular expressions for the vector and the gradient tensor of gravitation in a geocentric spherical frame, *Computers & Geosciences*, *34*, 1762-1768.

Eshagh, M. (2009). Alternative expressions for gravity gradients in local north-oriented frame and tensor spherical harmonics, *Acta Geophysica*, *58*, 215-243.

Eshagh, M., & Sjöberg, L. E. (2009). Topographic and atmospheric effects on GOCE gradiometric data in a local north-oriented frame: a case study in Fennoscandia and Iran, *Studia Geophysica et Geodaetica*, *53*, 61-80.

Fantino, E., & Casotto, S. (2009). Methods of harmonic synthesis for global geopotential models and their first, second and third order gradients, *Journal of Geodesy*, *83*, 595-619.

Fukushima, T. (2012a). Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating point numbers: II first-, second-, and third-order derivatives, *Journal of Geodesy*, *86*, 1019-1028.

Fukushima, T. (2012b). Numerical computation of spherical harmonics of arbitrary degree and order by extending exponent of floating point numbers, *Journal of Geodesy*, *86*, 271-285.

Heiskanen, W. A., & Moritz, H. (1967). *Physical Geodesy*, Freeman, SanFrancisco.

Hemant, K., & Maus, S. (2005). Geological modeling of the new CHAMP magnetic anomaly maps using a geographical information system technique, *Journal of Geophysical Research*, *110*, B12103, doi:10.1029/2005JB003837.

Hirt, C., Marti, U., Bürki, B., & Featherstone, W. E. (2010). Assessment of EGM2008 in Europe using accurate astrogeodetic vertical deflections and omission error estimates from SRTM/DTM2006.0 residual terrain model data, *Journal of Geophysical Research*, *115*, B10404, doi:10.1029/2009JB007057.

Holmes, S. A., & Featherstone, W. E. (2002a). A unified approach to the Clenshaw summation and the recursive computation of very high degree and order normalized associated Legendre functions, *Journal of Geodesy*, *76*, 279-299.

Holmes, S. A., & Featherstone, W. E. (2002b). Extending simplified high degree synthesis methods to second latitudinal derivatives of geopotential, *Journal of Geodesy*, *76*, 447-450.

Hotine, M., & Morrison, F. (1969). First Integrals of the equations of satellite motion, *Journal of Geodesy*, *43*, 41-45.

Huang, C.-L., Dehant, V., Liao, X.-H., Van Hoolst, T., & Rochester, M. G. (2011). On the coupling between magnetic field and nutation in a numerical integration approach, *Journal of Geophysical Research*, *116*, B03403, doi:10.1029/2010JB007713.

Ilk, K. H. (1983). *Ein beitrag zur dynamik ausgedehnter körper-gravitationswechselwirkung*, Deutsche Geodätische Kommission, Reihe C, Heft Nr.288, Muchen.

Jekeli, C., & Lee, J. K. (2007). On the computation and approximation of ultra-high-degree spherical harmonic series, *Journal of Geodesy*, *81*, 603-615.

Kim, J. W., Hwang, J. S., von Frese, R. R. B., Kim, H. R., & Lee, S.-H. (2007). Geomagnetic field modeling from satellite attitude control magnetometer measurements, *Journal of Geophysical Research*, *112*, B05105, doi:10.1029/2005JB004042.

Kotsiaros, S., & Olsen, N. (2012). The geomagnetic field gradient tensor: properties and parametrization in terms of spherical harmonics, *International Journal on Geomathematics*, *3*, 297-314.

Liu, X. G., Wu, X. P., & Wang, K. (2012). Construction of least squares collocation models for single component and composite components of disturbed gravity gradients, *Chinese Journal of Geophysics*, *55*, 1572-1580. (in Chinese)

Liu, X. G., Wu, X. P., Zhao, D. M., & Wu, X. (2010). Non-singular expression of the disturbing gravity gradients, *Geodaetica et Cartographica Sinica*, *39*, 450-457. (in Chinese)

Liu, X. G., Zhang, Y. F., Li, Y., & Xu, K. (2013). Construction of nonsingular formulae of variance and covariance function of disturbing gravity gradient tensors, *Geodesy and Geodynamics*, *4*, 1-8.

Liu, X. G., Sun, Z. M., Zhai, Z. H., Guan, B., Duan, W. C., Ma, J., & Qin, X. P. (2019). Non-singular spherical harmonic expressions of geomagnetic gradient tensors, *IUGG 2019*, 8-18

July 2019, Montréal, Canada.

Malin, S. R. C., & Pocock, S. B. (1969). Geomagnetic spherical harmonic analysis, *Pure and Applied Geophysics*, *75*, 117-132.

Moritz, H. (1980). *Advanced physical geodesy*, Abacus Press, Harleysville.

Pail, R., Bruinsma, S., Migliaccio, F., Förste, C., Goiginger, H., Schuh, W. D., Höck, E., Reguzzoni, M., Brockmann, J. M., Abrikosov, O., Veicherts, M., Fecher, T., Mayrhofer, R., Krasbutter, I., Sansò, F., & Tscherning, C. C. (2011). First GOCE gravity field models derived by three different approaches, *Journal of Geodesy*, *85*, 819-843.

Pavlis, N. K., Holmes, S. A., Kenyon, S. C., & Factor, J. K. (2012). The development and evaluation of the Earth Gravitational Model 2008 (EGM2008), *Journal of Geophysical Research*, *117*, B04406, doi:10.1029/2011JB008916.

Petrovskaya, M. S., & Vershkov, A. N. (2006). Non-Singular expressions for the gravity gradients in the local north-oriented and orbital reference frames, *Journal of Geodesy*, *80*, 117-127.

Petrovskaya, M. S., & Vershkov, A. N. (2007). Local orbital derivatives of the Earth potential expressed in terms of the satellite cartesian coordinates and velocity, *Artificial Satellites*, *42*, 17-39.

Petrovskaya, M. S., & Vershkov, A. N. (2008). Development of the second-order derivatives of the Earth's potential in the local north-oriented reference frame in orthogonal series of modified spherical harmonics, *Journal of Geodesy*, *82*, 929-944.

Quinn, J. M., Kerridge, D. J., & Barraclough, D. R. (1986). World magnetic charts for 1985 - spherical harmonic models of the geomagnetic field and its secular variation, *Geophysical Journal of the Royal Astronomical Society*, *87*, 1143-1157.

Ravat, D., Langel, R. A., Purucker, M., Arkani-Hamed, J., & Alsdorf, D. E. (1995). Global vector and scalar Magsat magnetic anomaly maps, *Journal of Geophysical Research*, *100*, B10, 20111-20136.

Rummel, R., Yi, W. Y., & Stummer, C. (2011). GOCE gravitational gradiometry, *Journal of Geodesy*, *85*, 777-790.

Shao, J. C., Fuller, M., Tanimoto, T., Dunn, J. R., & Stone, D. B. (1999). Spherical harmonic analyses of paleomagnetic data: the time-averaged geomagnetic field for the past 5 myr and the Brunhes-Matuyama reversal, *Journal of Geophysical Research*, *104*, 5015-5030.

Šprlák, M., & Novák, P. (2017). Spherical integral transforms of second-order gravitational tensor components onto third-order gravitational tensor components, *Journal of Geodesy*, *91*, 167-194.

Wan, X. Y. (2011). New derivation of nonsingular expression for gravitational gradients calculation, *Geomatics and Information Science of Wuhan University*, 36, 1486-1489. (in Chinese)

Wan, X. Y., & Yu, J. H. (2013). Derivation of the radial gradient of the gravity based on non-full tensor satellite gravity gradients, *Journal of Geodesy*, *66*, 59-64.

Wardinski, I., & Holme, R. (2006). A time-dependent model of the Earth's magnetic field and its secular variation for the period 1980-2000, *Journal of Geophysical Research*, *111*, B12101, doi:10.1029/2006JB004401.

Zhu, Y. C., Wan, X. Y., & Yu, J. H. (2017). Non-singular formulas for computing gravity vector and vertical deviation, *Geomatics and Information Science of Wuhan University*, *42*, 1854-1860.

(in Chinese)