Eigenvalue Uncoupling of Electrokinetic Flows

Kristopher Kuhlman[1] and Bwalya Malama[1]

[1]Sandia National Laboratories

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Abstract

We present an approach to uncoupling the pair of transient governing equations used in electrokinetics (i.e., streaming potential and electroosmosis). This approach allows for the solution of two uncoupled "intermediate" equations, then the physical solution is found by recombination of these intermediate potentials through a matrix multiplication. We present numerically stable expressions for the coefficients, and an example showing electrokinetics arising from pumping a fully penetrating well in a confined aquifer, surrounded by insulating aquicludes. Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525. (SAND2019-8712 A)

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Kristopher L. Kuhlman[(1)], Bwalya Malama[(2)]

1) Sandia National Laboratories, Albuquerque NM
2) California Polytechnic State University, San Luis Obispo CA

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There is a need to solve coupled physics problems efficiently. We present a simple approach to uncoupling electrokinetic flow using a matrix technique (Lo et al., 2009). Electrokinetics is movement of ions and water in a porous medium under fluid pressure (p) and electrostatic potential (ψ) gradients:

$$\begin{bmatrix} C^* & 0 \\ 0 & nc \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} \psi \\ p \end{bmatrix} = \begin{bmatrix} \sigma_0 & L_{12} \\ L_{21} & \frac{k_0}{\mu} \end{bmatrix} \nabla^2 \begin{bmatrix} \psi \\ p \end{bmatrix}$$

Consolidating coefficients leads to:

$$\frac{\partial}{\partial t} \begin{bmatrix} \psi \\ p \end{bmatrix} = \begin{bmatrix} \alpha_E & \alpha_E K_S \\ \alpha_H K_E & \alpha_H \end{bmatrix} \nabla^2 \begin{bmatrix} \psi \\ p \end{bmatrix}$$

Non-dimensionalizing the equations reduces free parameters from 4 to 2, the diffusivity ratio ($\alpha_D = \alpha_E/\alpha_H$) and the electrokinetic coupling strength ($K_D = K_E K_S$):

$$\frac{\partial}{\partial t_D} \underbrace{\begin{bmatrix} \psi_D \\ p_D \end{bmatrix}}_{\mathbf{d}_D} = \overbrace{\begin{bmatrix} \alpha_D & \alpha_D \\ K_D & 1 \end{bmatrix}}^{\boldsymbol{A}} \nabla_D^2 \begin{bmatrix} \psi_D \\ p_D \end{bmatrix}$$

Using the eigenvalue decomposition from linear algebra:

$$\left.\begin{aligned} \frac{\partial \mathbf{d}_D}{\partial t} &= \mathbf{A}\nabla_D^2 \mathbf{d}_D \\ \mathbf{A} &= \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1} \end{aligned}\right\} \frac{\partial}{\partial t_D} \underbrace{\mathbf{S}^{-1}\mathbf{d}_D}_{\boldsymbol{\delta}} = \overbrace{(\mathbf{S}^{-1}\mathbf{S})}^{\boldsymbol{I}} \boldsymbol{\Lambda} \nabla_D^2 \underbrace{\mathbf{S}^{-1}\mathbf{d}_D}_{\boldsymbol{\delta}}$$

We get two uncoupled governing equations (diagonal $\boldsymbol{\Lambda}$ matrix), but in terms of a vector of new potentials ($\boldsymbol{\delta} = \mathbf{S}^{-1}\mathbf{d}_D$). Eigenvector matrices ($\mathbf{S}$) and eigenvalue matrix ($\boldsymbol{\Lambda}$) are below.

Uncoupling Procedure:

1) Pose problem in terms of δ_1, δ_2
2) Solve 2 uncoupled diffusion problems
3) Compute p, ψ via matrix multiply ($\mathbf{d}_D = \mathbf{S}\boldsymbol{\delta}$)

$$\mathbf{S} = \begin{bmatrix} \frac{2\alpha_D}{1-\alpha_D-\Delta} & -\frac{1}{2K_D}(1-\alpha_D-\Delta) \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \frac{2\alpha_D(1-K_D)}{1+\alpha_D+\Delta} & 0 \\ 0 & \frac{1}{2}(1+\alpha_D+\Delta) \end{bmatrix}$$

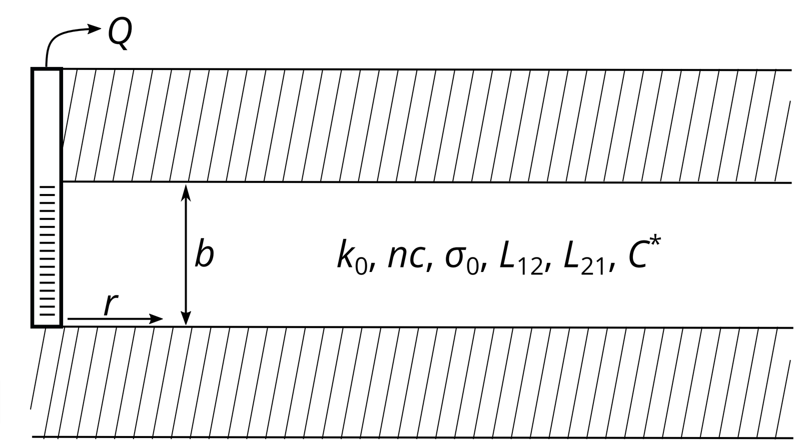
$$\mathbf{S}^{-1} = \frac{1}{\Delta} \begin{bmatrix} -K_D & -\frac{1}{2}(1-\alpha_D-\Delta) \\ K_D & \frac{-2K_D\alpha_D}{1-\alpha_D-\Delta} \end{bmatrix}$$

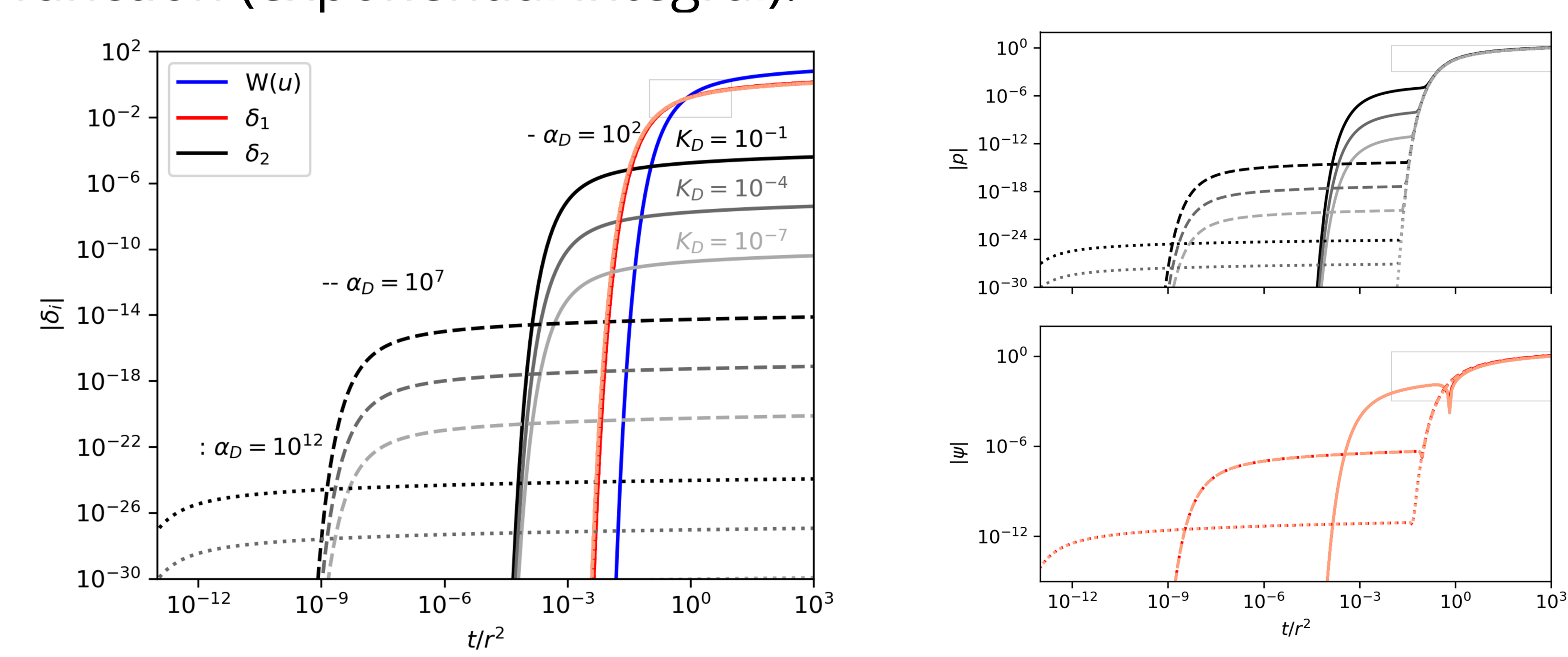
$$\Delta = \sqrt{1+\alpha_D(4K_D - 2 + \alpha_D)}$$

Lo, Sposito & Majer, 2009. Analytical decoupling of poroelasticity equations for acoustic-wave propagation and attenuation in a porous medium containing two immiscible fluids. *Journal of Engineering Mathematics*, 64:219–235.

Example 1. Flow to Pumping Well (Theis)

1D radial flow to a confined well surrounded by insulators for electrokinetic problem. The solution to the decoupled intermediate equations (δ_1, δ_2) comes from the Theis (1935) well function (exponential integral).





Initial type curve (blue), δ_1 solutions (red), and δ_2 solutions (black) for a range of α_D and K_D (left). The solution for (p, ψ) is recombined (right) using $\mathbf{d}_D = \mathbf{S}\boldsymbol{\delta}$.

Using only *type curve* methods, we solve coupled streaming potential and electroosmosis.

x-axis: $\overbrace{\ln\left[\frac{t}{r^2}\right]}^{\text{data}} = \overbrace{\ln\left[\frac{1}{4\Lambda_{ii}}\right]}^{\text{shift}} + \overbrace{\ln\left[\frac{1}{u}\right]}^{\text{type-curve}}$

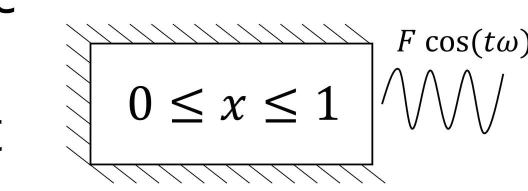
y-axis: $\ln[\delta_i] = \ln\left[\frac{Q_i}{4\pi\Lambda_{ii}}\right] + \ln[\mathrm{E}_1(u)]$

Variable	Description	Dimensions
α_H, α_E	hydraulic/electrical diffusivity	$\mathrm{m^2/sec}$
t	time	sec
k_0	formation permeability	$\mathrm{m^2}$
c	formation compressibility	1/Pa
n	formation porosity	–
C^*	electric capacitance of formation	$\mathrm{C/(m^3 \cdot V)}$
σ_0	electric conductivity	S/m
p	change in fluid pressure	Pa
ψ	change in electrostatic potential	V
μ	viscosity of water	$\mathrm{Pa \cdot sec}$
L_{12}, L_{21}	electrokinetic coupling coefficient	$\mathrm{A/(Pa \cdot m) = m^2/(V \cdot sec)}$
K_S	streaming potential	V/Pa
K_E	electroosmosis pressure	Pa/V

Theis, 1935. The relation between lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage. *Transactions, American Geophysical Union*, 16(2):519–524.

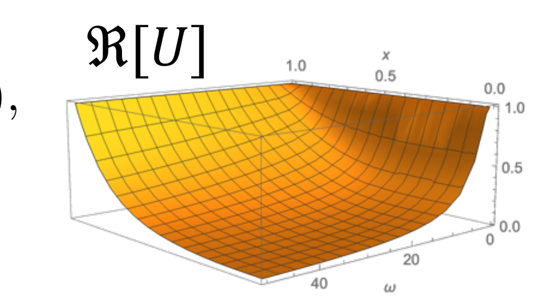
Example 2. Driven Electrokinetic Core

1D flow in a box driven by periodic Dirichlet BC on one end, no-flow / insulating on other end (Pengra et al., 1999). Steady-state solution is:



$$\delta_i(x_D, t_D) = \delta_{i\mathrm{SS}}(x_D, t_D) + \cancel{\delta_{i\mathrm{TR}}(x_D, t_D)}^{\,0},$$

$$\delta_{i\mathrm{SS}}(x_D, t_D) = \Re\left[U(x_D) e^{j\omega_D t_D}\right]$$



Solve ODE for complex amplitude (U), given BC

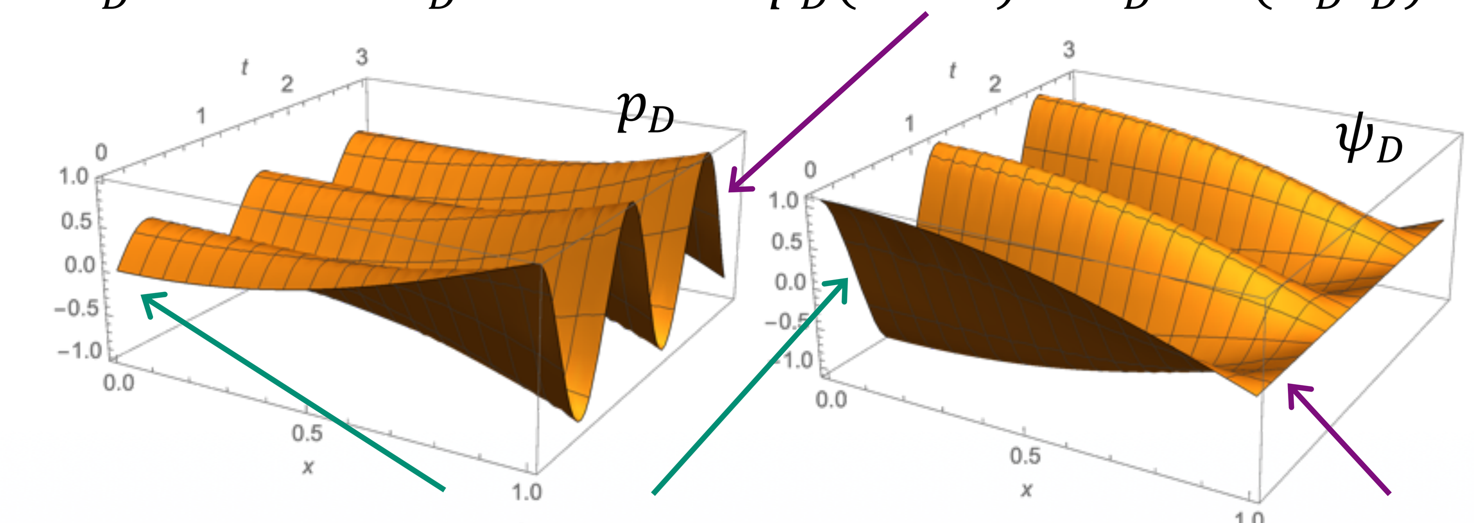
$$\frac{\mathrm{d}^2 U}{\mathrm{d}x_D^2} - \frac{j\omega_D}{\Lambda_{ii}} U = 0 \qquad U(x_D) = c_1 e^{\zeta x_D} + c_2 e^{-\zeta x_D} \qquad \zeta = \sqrt{j\omega_D/\Lambda_{ii}}$$

$$\delta_{i\mathrm{SS}}(x_D, t_D) = \Re\left\{\frac{jK_D F_D \zeta}{\omega_D \Delta\left(e^{\zeta} - 1\right)}\left[e^{i\omega_D t_D + \zeta(1+x_D)} + e^{i\omega_D t_D - \zeta x_D}\right]\right\}$$

Solution is applied to electrokinetics via decoupling

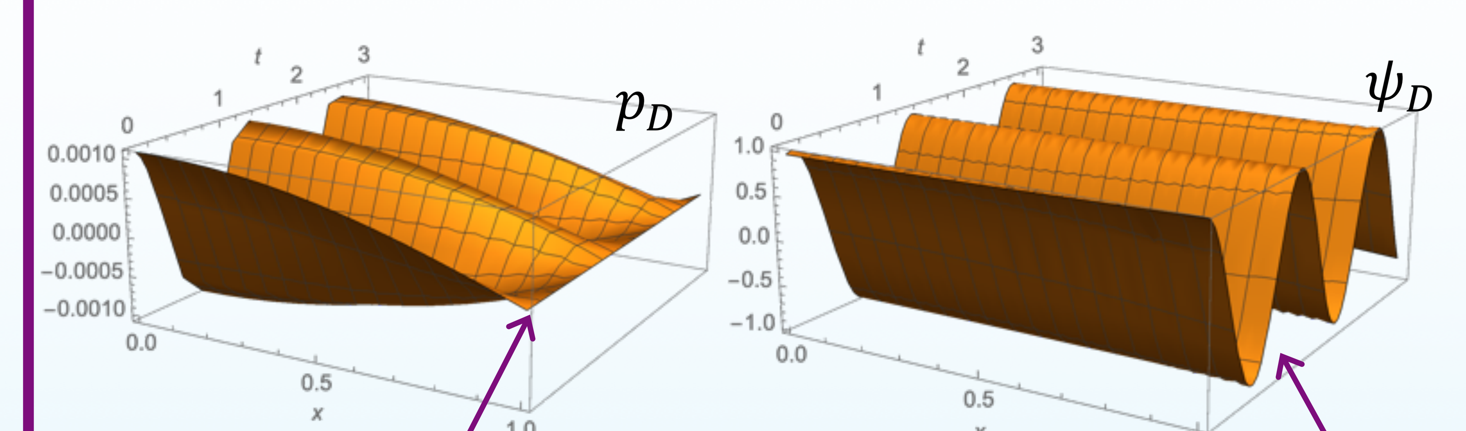
Applied Pressure BC (Streaming Potential)

$\alpha_D = 10$ $\quad K_D = 0.01$ $\quad p_D(x=1) = F_D\cos(\omega_D t_D)$



No-gradient BC ($x = 0$) $\quad \psi_D(x=1) = 0$

Applied Voltage BC (Electroosmosis)



$\psi_D(x=1) = 0$ $\quad \psi_D(x=1) = F_D\cos(\omega_D t_D)$

Pengra, Li & Wong, 1999. Determination of rock properties by low-frequency AC electrokinetics. *Journal of Geophysical Research*, 104(B12):29485-29508.

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