Fractional relaxation noises, motions and the fractionalenergy balance equation

Shaun Lovejoy[1]

[1]McGill University

November 24, 2022

Abstract

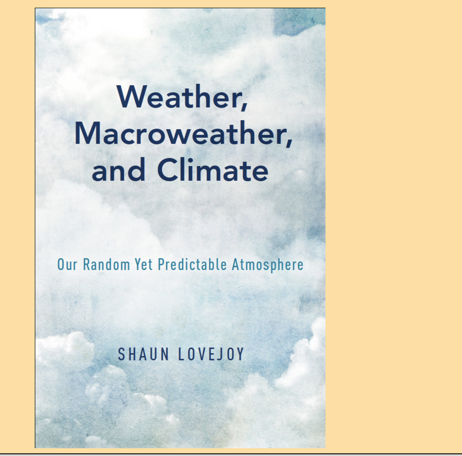
We consider the statistical properties of solutions of the stochastic fractional relaxation equation that has been proposed as a model for the earth's energy balance. In thisequation, the (scaling) fractional derivative term modelsenergy storage processes that occur over a wide range of space and time scales. Up until now, stochastic fractionalrelaxation processes have only been considered withRiemann-Liouville fractional derivatives in the context of random walk processes where it yields highlynonstationary behaviour. For our purposes we require the stationary processes that are the solutions of the Weyl fractional relaxation equations whose domain is -[?] to t rather than 0 to t. We develop a framework for handling fractional equationsdriven by white noise forcings. To avoid divergences, wefollow the approach used in fractional Brownian motion(fBm). The resulting fractional relaxation motions (fRm) and fractional relaxation noises (fRn) generalize the more familiar fBm and fGn (fractional Gaussian noise). Weanalytically determine both the small and large scale limitsand show extensive analytic and numerical results on the autocorrelation functions, Haar fluctuations and spectra. We display sample realizations. Finally, we discuss the prediction of fRn, fRm which – due to long memories - is a past value problem, not an initial value problem. We develop an analytic formula for the fRnforecast skill and compare it to fGn. Although the large scale limit is an (unpredictable) white noise that is attainedin a slow power law manner, when the temporal resolutionof the series is small compared to the relaxation time, fRncan mimick a long memory process with a wide range of exponents ranging from fGn to fBm and beyond. Wediscuss the implications for monthly, seasonal, annualforecasts of the earth's temperature.

Abstract ID: 515303

The Fractional and Half-Order Energy Balance Equations

S. Lovejoy

Physics Department, McGill University, 3600 University st., Montreal, Que. H3A 2T8, Canada



Fractional relaxation noises, motions

The Fractional order Energy Balance Equation (FEBE)

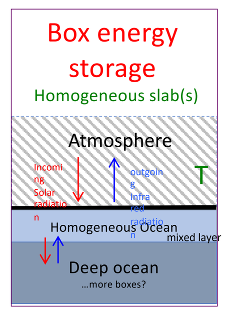
The (conventional) Energy Balance Equation (EBE)

$$S(t)+E(t)=E_{\downarrow}(t)$$

Total energy stored; Total energy emitted to outer space; Total energy from sun, volcanoes, outside system

$$S=CT \qquad E=\lambda^{-1}\int T\,dt \qquad E_{\downarrow}=\int F\,dt$$

Specific heat of the box; Climate Sensitivity; Linearized black body radiation leaving earth; Forcing



Differentiate: $C\frac{dT}{dt}+\lambda^{-1}T=F$

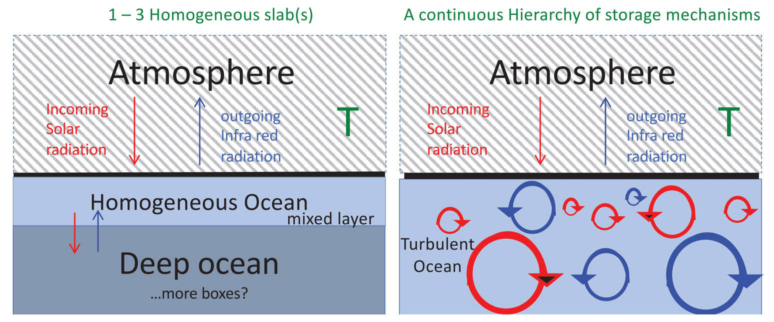
Box model: Newton's law of cooling

Time constant of box: $\tau=C\lambda$

Box storage | Scaling storage

IPCC AR5 recommends two boxes with τ_1 =8.4 yrs, τ_2 =410 yrs

1 – 3 Homogeneous slab(s) | A continuous Hierarchy of storage mechanisms



Scaling of the storage

Total storage with scaling parameter H

$$S(t)=E_{\downarrow}(t)-E_{\uparrow}(t)$$

$$S_H(t)=\frac{C_H}{\Gamma(1-H)}\int_{-\infty}^{t}T(s)\left(\frac{t-s}{\tau}\right)^{-H}\frac{ds}{\tau};\quad 0\le H\le 1$$

(Weyl) fractional integral order H-1; Relaxation time

....Storage depends on the past: long memory

Classical box result: limit $H\to 1$:

$$S_1(t)=C_1T(t)$$

Energy rate equation

$$S(t)=E_{\downarrow}(t)-E_{\uparrow}(t)$$

Differentiate: $\frac{dS_H(t)}{dt}+\lambda^{-1}T(t)=F(t)$

The FEBE

$$\tau^{H}\,{}_{-\infty}D_t^{H}T+T=\lambda F \qquad \tau=\lambda C_H$$

Fourier space: $\left((i\omega\tau)^H+1\right)\widehat{T}=\lambda\widehat{F}$

Definition of the H order Weyl fractional derivative

Fractional Generalization: realistic storage

H = 1 | 0≤H≤1

$$\tau\frac{dT}{dt}+T=\lambda F \quad \text{(EBE)} \qquad \xrightarrow{\text{Fractional Generalization}} \qquad \tau^{H}\frac{d^{H}T}{dt^{H}}+T=\lambda F \quad \text{(FEBE)}$$

Energy Balance Equation | Fractional Energy Balance Equation

Solution $T(t)=\lambda G_H * F'(t)$ (Convolution; Step function response)

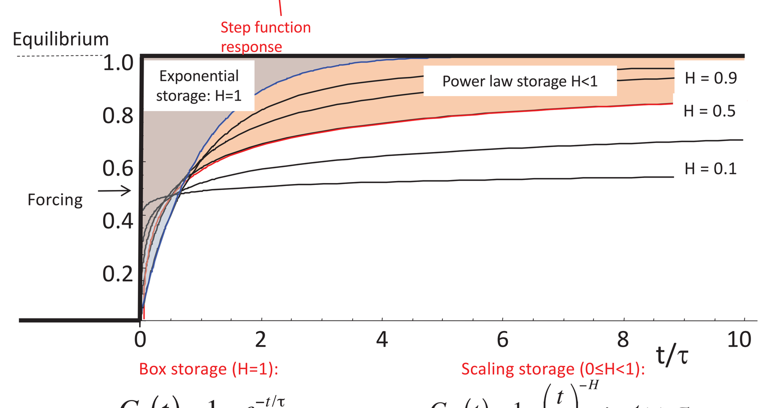
The FEBE assumes scaling of the storage processes, not scaling of the temperature response...

$$G_1(t)=1-e^{-t/\tau};\quad H=1 \qquad \xrightarrow{\text{Fractional Generalization}} \qquad G_H(t)=\left(\frac{t}{\tau}\right)^{H}E_{H,H+1}\left(-\left(\frac{t}{\tau}\right)^{H}\right);\quad 0\le H<1$$

Exponential | Generalized exponential

Response: Generalized relaxation processes

$$T(t)=\lambda\int G_H(t-t')F'(t')dt';\quad F'(t)=\frac{dF}{dt}$$



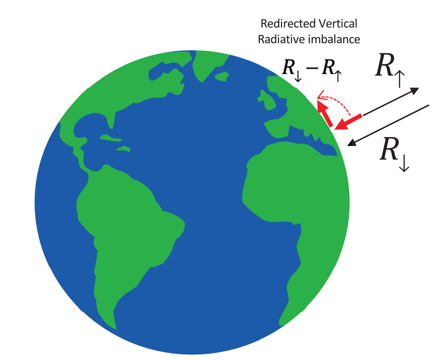
Box storage (H=1): $G_1(t)=1-e^{-t/\tau}$ | Scaling storage (0<H<1): $G_H(t)\approx 1-\left(\frac{t}{\tau}\right)^{-H};\quad t>>\tau$

The Half-order Energy Balance Equation (HEBE)

The conventional 1-D energy balance equation (EBE) has no vertical coordinate so that radiative imbalances between the earth and outer space are redirected in the horizontal in an ad hoc manner. We retain the basic EBE but add a vertical coordinate so that the imbalances drive the system by imposing heat fluxes through the surface. While this is theoretically correct, it leads to (apparently) difficult mixed boundary conditions. However using Babenko's method, we directly obtain simple equations for (2D) surface temperature anomalies $T_s(\underline{x},t)$: the Half-order Energy Balance Equation (HEBE) and the Generalized HEBE, (GHEBE). The HEBE anomaly equation only depends on the local climate sensitivities and relaxation times. We analytically solve the HEBE and GHEBE for $T_s(\underline{x},t)$ and provide evidence that the HEBE applies at scales >≈10km. We also calculate very long time diffusive transport dominated climate states as well as space-time statistics including the cross-correlation matrix needed for empirical orthogonal functions.

The HEBE is the special H = 1/2 case of the Fractional EBE (FEBE) and has a long (power law) memory up to its relaxation time t. Several papers have empirically estimated H ≈ 0.5, as well as t ≈ 4 years, whereas the classical zero-dimensional EBE has H = 1 and t ≈ 4 years. The former values permit accurate macroweather forecasts and low uncertainty climate projections; this suggests that the HEBE could apply to time scales as short as a month. Future generalizations include albedo-temperature feedbacks and more realistic treatments of past and future climate states.

Regional energy balance Budyko-Sellers

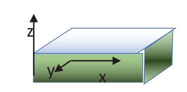


$$\underline{Q}_c(y)=\left(\rho c\left(v_yT_c-\kappa_y\frac{\partial\overline{T}_c}{\partial y}\right)+sign(y)\left(R_{\downarrow}(y)-R_{\uparrow}(\overline{T}_c)\right)\right)\hat{y}$$

Average Meridional Heat Flux; Vertical flux imbalance

$$\nabla\cdot\underline{Q}_c=0$$

Correcting Budyko-Seller's boundary conditions: H=1/2 - the Half-Order Energy Balance Equation (HEBE)



(homogeneous) 3-D Energy equation: $T(\underline{x},z,t)$

$$\frac{\partial T}{\partial t}+\underline{v}\cdot\nabla_hT-\kappa_h\nabla_h^2T-\kappa_v\frac{\partial^2T}{\partial z^2}=0$$

Mixed boundary condition, (vertical heat fluxes)

$$\left.\frac{T(\underline{x},z,t)}{\lambda}+\rho c\kappa_v\frac{\partial T(\underline{x},z,t)}{\partial z}\right|_{z=0}=F(\underline{x},t)$$

Outgoing radiative flux; Climate sensitivity; Heat conduction through the surface (forcing); External forcing

2-D (surface) Energy equation for $T_s(\underline{x},t)= T(\underline{x},0,t)$

Scales >> 10 km

HEBE $\quad \tau(\underline{x})^{1/2}\,{}_{-\infty}D_t^{1/2}T_s+T_s=\lambda(\underline{x})F(\underline{x},t)$

Babenko's method

$$\frac{\partial T}{\partial t}+\underline{v}\cdot\nabla_hT-\kappa_h\nabla_h^2T-\kappa_v\frac{\partial^2T}{\partial z^2}=0 \qquad T=T(\underline{x},y,z)$$

Nondimensional form

$$\left(\tau\frac{\partial}{\partial t}+l_h\zeta-l_v^2\frac{\partial^2}{\partial z^2}\right)T=0;\quad \zeta=\left(\frac{\underline{v}}{\kappa_h}\cdot\nabla_h-l_h\nabla_h^2\right)$$

Factor: $\left(L+l_v\frac{\partial}{\partial z}\right)\left(L-l_v\frac{\partial}{\partial z}\right)T=0;\quad L=\left(\tau\frac{\partial}{\partial t}+l_h\zeta\right)^{1/2}$

Relaxation time; Vertical diffusion length; Horizontal advection; Horizontal diffusion length

Fractional relaxation noises, motions and the FEBE

We consider the statistical properties of solutions of the stochastic fractional relaxation equation that has been proposed as a model for the earth's energy balance. In this equation, the (scaling) fractional derivative term models energy storage processes that occur over a wide range of space and time scales. Up until now, stochastic fractional relaxation processes have only been considered with Riemann-Liouville fractional derivatives in the context of random walk processes where it yields highly nonstationary behaviour. For our purposes we require the stationary processes that are the solutions of the Weyl fractional relaxation equations whose domain is to t rather than 0 to t.

We develop a framework for handling fractional equations driven by white noise forcings. To avoid divergences, we follow the approach used in fractional Brownian motion (fBm). The resulting fractional relaxation motions (fRm) and fractional relaxation noises (fRn) generalize the more familiar fBm and fGn (fractional Gaussian noise). We analytically determine both the small and large scale limits and show extensive analytic and numerical results on the autocorrelation functions, Haar fluctuations and spectra. We display sample realizations.

Finally, we discuss the prediction of fRn, fRm which – due to long memories is a *past* value problem, not an *initial* value problem. We develop an analytic formula for the fRn forecast skill and compare it to fGn. Although the large scale limit is an (unpredictable) white noise that is attained in a slow power law manner, when the temporal resolution of the series is small compared to the relaxation time, fRn can mimic a long memory process with a wide range of exponents ranging from fGn to fBm and beyond. We discuss the implications for monthly, seasonal, annual forecasts of the earth's temperature as well as for projecting the temperature to 2050 and 2100.

fRn, fRm

The stochastic FEBE, fractional Relaxation motion (fRm), fractional Relaxation noise (fRn)

In the previous section we considered the deterministic response to a simple piecewise linear model of external forcing. We noted that since the FEBE is a linear equation that we could separately model this deterministic response and the stochastic, internal variability. In this section therefore we turn to the pure stochastic case driven by white noise "innovations", we briefly summarize some of the results in [*Lovejoy*, 2019a].

The physical problem that we wish to solve is the noise driven FEBE with noise amplitude σ and sensitivity λ:

$$\tau^{H}\,{}_{-\infty}D_t^{H}T_i+T_i=\lambda f(t);\quad f(t)=\sigma\gamma(t) \qquad (1)$$

with initial conditions $T_i(-\infty)=0$, 0≤H≤1 and γ(t) a unit Gaussian white noise so that $\left\langle f^2\right\rangle^{1/2}=\sigma;\ \left\langle f\right\rangle=0$, σ is the amplitude of the noise. To understand the statistical properties of the internal variability (the response), it suffices to study the nondimensional equation:

$${}_{-\infty}D_t^{H}U_H+U_H=K\gamma(t) \qquad (2)$$

where K is the normalization constant in eq. 35 and the nondimensional $U_H(t)$ function is called the fractional Relaxation noise (fRn) since it generalizes fGn. Using U_H we can obtain the solution to the dimensional eq. 46:

$$T(t)=\frac{\lambda\sigma}{K\tau}U_H\left(\frac{t}{\tau}\right) \qquad (3)$$

Since the fRn process is the solution of the fractional relaxation equation with a stationary, Gaussian, zero mean, white noise forcing, it is also stationary, Gaussian with zero mean. Its statistics are therefore fully characterized by its autocorrelation function.

We can now calculate the correlation function relevant for the fRn statistics. The main complication is that in the small t limit, the fractional term dominates so that we obtain the (fGn) limit. The solution of eq. 47 is therefore - like γ(t) - a generalized function: to obtain solutions with finite variances, we must take averages over finite resolutions τ_w.

The resulting τ_w resolution autocorrelation function at lag Δt is:

$$R_{\tau_w}(\Delta t)=\left\langle T_{\tau_w}(t)T_{\tau_w}(t-\Delta t)\right\rangle=\int_0^{\infty}G_{0,H}(\Delta t+t')G_{0,H}(t')dt';\quad t>\tau_w$$
$$R_{\tau_w}(0)=\left\langle T_{\tau_w}(t)^2\right\rangle=\tau_w^{2H-1} \qquad (4)$$

[*Lovejoy*, 2019a], where we used the normalization K given above. From this, we can obtain the high frequency fGn approximation (valid for Δt<<1 corresponding to t<<τ in the dimensional equation):

$$R_{H,\tau_w}(\Delta t)\approx\frac{1}{2}\tau_w^{2H-1}\left((\lambda+1)^{2H+1}+(\lambda-1)^{2H+1}-2\lambda^{2H+1}\right);\quad \lambda=\Delta t/\tau_w;\quad \tau_w\le\Delta t<<1;\quad 0<H\le\frac{1}{2}$$
$$R_{H,\tau_w}(0)=\tau_w^{2H-1} \qquad (5)$$

or approximately:

$$R_{H,\tau_w}(\Delta t)\approx H(2H+1)\Delta t^{2H-1};\quad \tau_w<<\Delta t<<1;\quad 0<H<\frac{1}{2} \qquad (6)$$

At low frequencies, for Δt>>1 (corresponding to Δt>>τ in the dimensional equation), we obtain:

$$R_H(\Delta t)=-\frac{K^2}{\Gamma(-H)}\Delta t^{-1-H}+O\left(\Delta t^{-1-2H}\right):\quad 0<H<2;\quad \Delta t>>1 \qquad (7)$$

Note that this result is independent of the resolution τ_w and that it holds over a wider range of H values. A technical point is that although the fGn has a huge memory (the slow Δt^{2H-1} fall-off in R(Δt)), the corresponding fRn is effectively truncated with a faster long time behaviour that decays as τ^{-1-H}.

We could also mention the total energy storage $S_H(t)$. From eq. ? the nondimensional rate of storage dS_H/dt = Kγ-U. The storage thus involves the integral of the fRn. The integral of fRn is "fractional Relaxation motion (fRm)", the name, in analogy with the integral of fGn which is called "fractional Brownian motion (fBm)" (see [*Lovejoy*, 2019a]). For long times, we have:

$$\left\langle\Delta S_H(\Delta t)^2\right\rangle=\frac{1}{(1-2H)\Gamma(1-H)^2}\Delta t^{1-2H};\quad \Delta S_H(\Delta t)=S_H(t)-S_H(t-\Delta t);\quad \Delta t>>1;\quad 0<H<1/2 \qquad (8)$$

which is a fractional Brownian motion (of order ½-H) in comparison the nondimensional temperature integral (proportional to the total energy emitted to outer space) increases as Δt for large Δt corresponding to a usual Brownian motion.

A final interesting property is the predictability of fRn. fGn has an enormous memory (it is called a "long range memory process") and since for short times, fRn is close to fGn the two can both be well predicted (for fGn, the skill –with infinite past data- becomes perfect in the limit H->1 (H_l->0)), however, due to the cutoff, fRn cannot be well predicted beyond the relaxation time τ.

Conclusions

References

Lovejoy, S., Fractional Relaxation noises, motions and the stochastic fractional relaxation equation *Nonlinear Proc. in Geophys. Disc.*, https://doi.org/10.5194/npg-2019-39, 2019a.
Lovejoy, S., *Weather, Macroweather and Climate: our random yet predictable atmosphere*, 334 pp., Oxford U. Press, 2019b.
Lovejoy, S., The half-order energy balance equation, *J. Geophys. Res. (Atmos.)*, *(submitted, Nov. 2019)*, 2019c.
Lovejoy, S., Procyk, R., del Rio Amador, L., and Hébert, R., The fractional Energy Balance Equation, *Earth Syst. Dyn. Disc.*, *(in preparation)*, 2019.

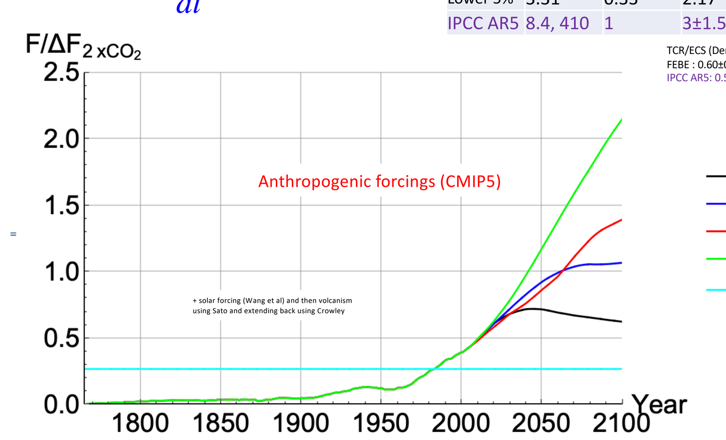
Projecting the temperature to 2100 using FEBE

Parameters

FEBE

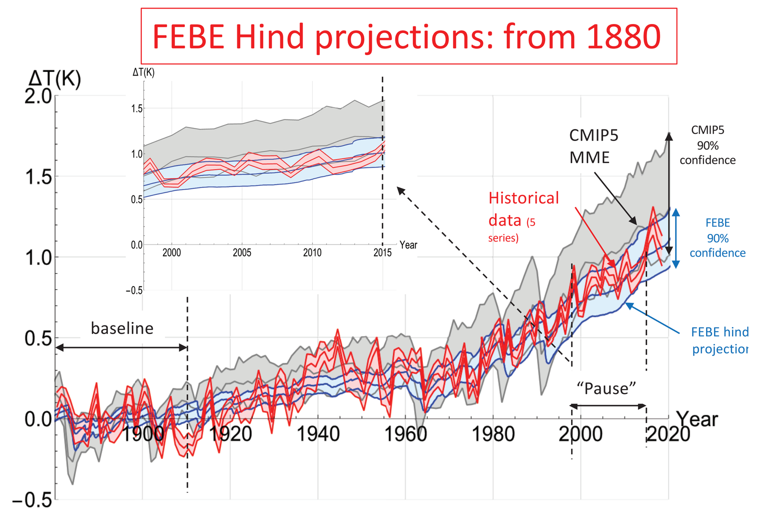
$$\tau^{H}\frac{d^{H}T}{dt^{H}}+T=\lambda F$$

	τ (yrs) Relaxation time	H	λ Climate sensitivity (°C/CO₂ doubling)
Upper 5%	6.70	0.40	2.68
Median	5.01	0.37	2.43
Lower 5%	3.31	0.33	2.17
IPCC AR5	8.4, 410	1	3±1.5

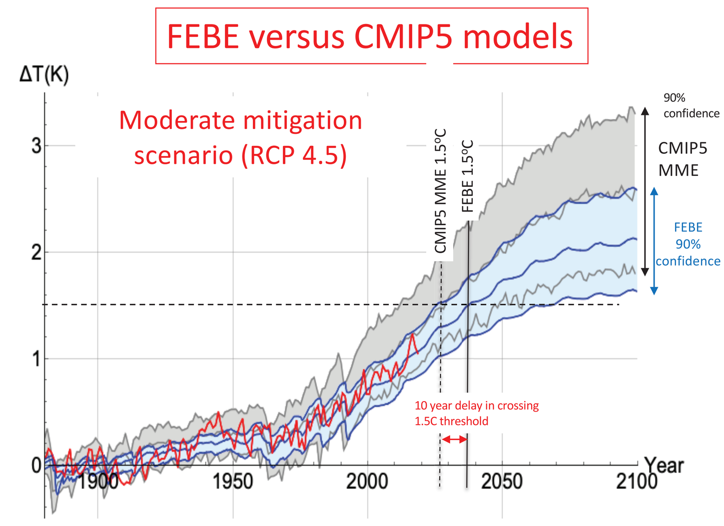


With R. Procyk

FEBE Hind projections: from 1880



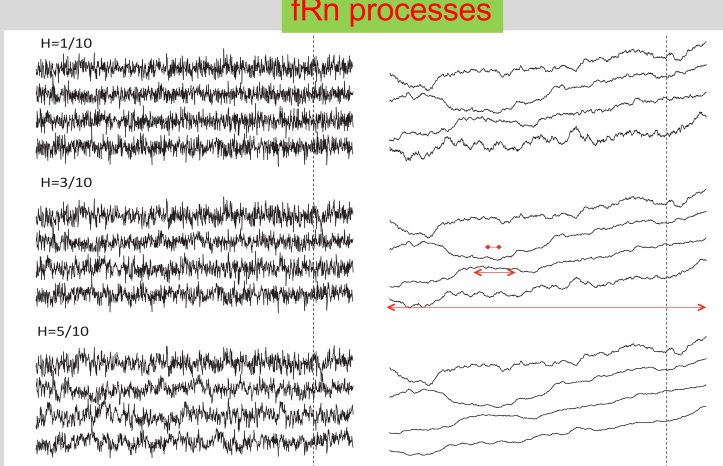
FEBE versus CMIP5 models



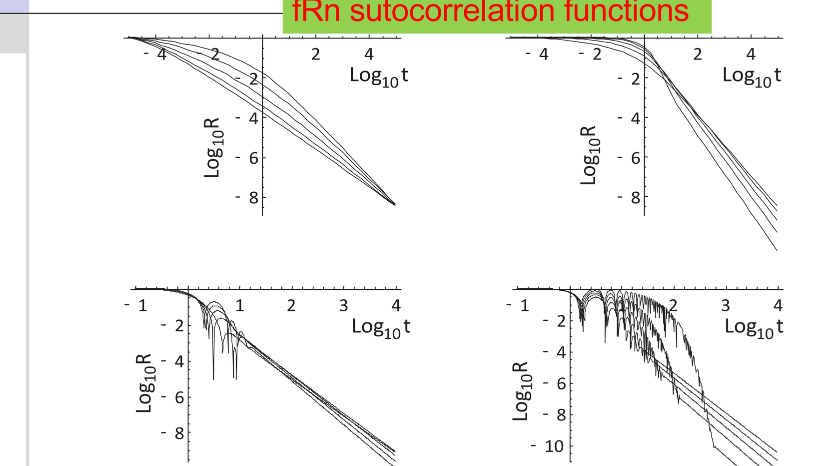
....Lower uncertainties

With Roman Procyk

fRn processes



fRn sutocorrelation functions



fRn predictability skill