Prescribed Time Convergence for Continuous Action Iterative Dilemma with Lyapunov Analysis

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***Abstract*—This paper proposes a new approach for convergence analysis of continuous action iterative dilemma (CAID) to reach a stable consensus outcome within the prescribed time. Unlike usual game theory, where players can only choose between two options, i.e., cooperation or defection, CAID lets players pick from varying options and make more nuanced decisions. In the proposed strategy, a prescribed function, which is tunable by the user, adapts the learning rate of the player's strategy. This new method guarantees that players will eventually agree on a single strategy, regardless of where they started initially, and it achieves this agreement in a predefined time set by the user. The Lyapunov analysis guarantees the convergence of strategy to a consensus within a prescribed time. The simulation results of the proposed scheme with two evolutionary game examples demonstrate faster convergence and fewer iterations compared to the state-of-the-art method.**

***Index Terms*—Prescribed time, Convergence analysis, Evolutionary game theory, Social dilemmas, Lyapunov theory.**

I. Introduction

Recent interest in social networks has spurred research into the evolution of cooperation among individuals, notably through the application of evolutionary game theory [1]. This mathematical framework is vital for analyzing how systems evolve and group behaviors adapt in changing environments [2]. Specifically, evolutionary game theory utilizes well-established models, such as the prisoner's dilemma and the snowdrift game, to explore these dynamics in depth [3]. These models have been the focus of extensive study and have yielded noteworthy findings [4]. However, traditional applications of evolutionary game theory often adopt a binary strategy framework, limiting players to choices of either absolute cooperation or outright defection. This dichotomy fails to capture the complexity of strategies observed in real-world interactions, where individuals' actions exhibit a wider range of subtlety and diversity. Therefore, adopting nuanced continuous strategies is critical to represent complex social interactions [5]. This paper explores the continuous action iterated dilemma (CAID), which allows for a spectrum of cooperative behaviors and offers a more realistic depiction of social dynamics [6]. CAID extends beyond the binary framework, providing new insights into the complexity of autonomous individuals' strategies in social networks.

Numerous methodologies have been developed to study the convergence properties of evolutionary game theory [7]. An investigation into an evolutionary network game model incorporating delays, presented in [8], focuses on the game's progression toward equilibrium and the emergence of stable strategies. The work in [9] introduces a centralized algorithm that coordinates groups of devices through the principles of evolutionary game theory. Additionally, the study in [10] reveals a dynamical framework for analytically examining the evolution of cooperation across complex networks. Furthermore, an exploration of global convergence related to replicator dynamics in scenarios filled with hybrid agents engaging in repeated snowdrift games is discussed in [11]. A common technique for affirming convergence in evolutionary gaming scenarios involves utilizing the Jacobian matrix. This approach exhibits a dependence on the network structure among players, which increases the struggles in analyzing the convergence behavior of large and complicated networks. Accordingly, developing novel methodologies to evaluate the convergence dynamics is a pragmatic pursuit with the evolutionary game theory.

The Lyapunov function is extensively employed in the analysis of stability and convergence for complex nonlinear dynamical systems [12]. The study in [13] utilizes the Lyapunov function as a superior method to analyze the asymptotic convergence of evolutionary dynamics within intricate networks. Furthermore, using the Lyapunov function, the work in [14] explores the asymptotic convergence analysis of evolutionary dynamics with two-layer networks. To preserve accelerated convergence time with further robustness against uncertainties, the finite-time stability is presented in [15]. Finite-time stability is characterized by the property that the required time for system trajectories to reach equilibrium points is confined to a finite duration. Finite-time convergence analysis for CAID is presented in [16], offering a nuanced perspective on player behavior and the dynamics of continuous actions. However, the finite time analysis revealed that the convergence time is influenced by the initial strategies of the players. As a result, strategies that start far away from the consensus value take a longer time to reach convergence. Recently, the prescribed-time stability has been introduced in [17], which guarantees an adjustable and accelerated convergence time regardless of the initial conditions. This approach achieves convergence at any pre-assigned time by directly setting the time parameter in the

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structure design [18]. To the best of the authors' knowledge, current literature in evolutionary game theory does not cover the application of prescribed-time convergence analysis.

This paper proposes a prescribed time convergence approach for the CAID problem with the following key contributions.

- This work introduces a new model for understanding cooperation in networks. Unlike the traditional binary approach that limits players to only two choices, the proposed CAID model allows for a wider range of strategies. This additional complexity provides a more accurate picture of how cooperation evolves over time.
- A novel analytical approach is introduced for assessing the convergence of evolutionary dynamics in CAID. The proposed prescribed time convergence method achieves a faster convergence time of CAID consensus. More importantly, this method attains convergence at any pre-assigned time, which is explicitly defined as a time parameter in the proposed strategy.
- The Lyapunov theory is employed for the prescribed time convergence analysis of CAID.
- The effectiveness of the proposed strategy is demonstrated through simulations involving two evolutionary games. A comparative analysis of these simulations strengthens the validity of the proposed approach.

The remainder of the paper is organized as follows: Section II discusses the problem formulation for the dynamic model of continuous action iterative dilemma. Section III presents the main result of the proposed scheme, where the prescribed time convergence analysis is established. Section IV illustrates the comparative numerical analysis for the proposed strategy with two evolutionary game examples. The conclusion is made in Section V.

Notation

Throughout this paper, $\mathbb{R}$ is a real number, $\mathbb{R}^n$ denotes n real vector, $\mathbb{R}^{n\times n}$ represents $n \times n$ real matrix. For $z \in \mathbb{R}$ and $\alpha > 0$, the symbol $\lfloor z \rceil^{\alpha} = |z|^{\alpha}\text{sign}(z) \in \mathbb{R}$, where $\text{sign}(\cdot)$ is a standard signum function and $|\cdot|$ is an absolute operator. The symbol z^T denotes the transpose of z, for $\boldsymbol{z} = [z_1, z_2, z_3, \ldots, z_n]^T \in \mathbb{R}^n$ the symbol $\|\boldsymbol{z}\| = \sqrt{\boldsymbol{z}^T\boldsymbol{z}}$, $\lambda_{\min 2}(\mathcal{L})$ represents the second smallest eigenvalue of $\mathcal{L} \in \mathbb{R}^{n\times n}$.

II. Problem Formulation

Consider a scenario with $\mathcal{N}$ players, and the connections between these players are determined by an adjacency matrix denoted as $\mathcal{A} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$. If there's a connection between player i and player j, then the element a_{ij} of $\mathcal{A}$ is 1, otherwise zero. Since this paper utilizes a fully connected network, $a_{ij} = a_{ji} = 1$ can be achieved for every set of i, j players. Unlike the binary strategy commonly used in conventional evolutionary game theory, this paper employs the CAID approach, where i^{th} player strategy is continuous, i.e., $z_i \in [0, 1]$. Note that full cooperation is represented by $z_i = 1$, while complete defection is represented by $z_i = 0$. The payoff matrix of a 2-player game under CAID can be represented by:

$$\begin{bmatrix} w_0 & w_1 \\ w_2 & w_3 \end{bmatrix} \tag{1}$$

where w_i for $i = 0, 1, 2, 3$, signifies the player's playoff. Based on Darwin's survival of the fittest theory, the strategy fitness between two players (i and j) can be defined by $\mathcal{F}(z_i, z_j)$ as:

$$\begin{aligned} \mathcal{F}(\cdot) &= w_0 z_i z_j + w_1(1-z_i)z_j + w_2 z_i(1-z_j) + w_3(1-z_i)(1-z_j) \\ &= (w_0 - w_1 - w_2 + w_3)z_i z_j + (w_1 - w_3)z_j \\ &\quad + (w_2 - w_3)z_i + w_3. \end{aligned} \tag{2}$$

The difference between the strategy fitness is evaluated as

$$\tilde{\mathcal{F}}_{ji} = \mathcal{F}(z_j, z_i) - \mathcal{F}(z_i, z_j) \tag{3}$$

Inspired by imitation dynamics, players choose one of their neighbors' strategies with a certain probability, leading to the following dynamics [16]:

$$z_i(\kappa + 1) = (1 - q_{ij})z_i(\kappa) + q_{ij} z_i(\kappa), \tag{4}$$

where κ represents iteration number, $q_{ij} = \mu/(1+\exp^{-\wp|\tilde{\mathcal{F}}_{ji}|})$, μ and $\wp$ are two positive constants. Using the difference equation $\Delta z_i(\kappa) = z_i(\kappa+1) - z_i(\kappa)$, the dynamic equation for strategy adaptation in a two-player CAID game can be expressed as

$$\dot{z}_i(t) = q_{ij}(z_j(t) - z_i(t)). \tag{5}$$

Given the interconnected relation among $\mathcal{N}$ players, the strategy fitness of player i ($\mathcal{F}(z_i)$) can be determined using (2) as:

$$\begin{aligned} \mathcal{F}(z_i) = \sum_{j=1}^{\mathcal{N}} \mathcal{F}(z_i, z_j) = \sum_{j=1}^{\mathcal{N}} ((w_0 - w_1 - w_2 + w_3)z_i z_j \\ + (w_1 - w_3)z_j + (w_2 - w_3)z_i + w_3). \end{aligned} \tag{6}$$

Therefore, the difference between the strategy fitness of player i and player j can be inferred as:

$$\tilde{\mathcal{F}_{ji}} = \mathcal{F}(z_j) - \mathcal{F}(z_i). \tag{7}$$

Like in dynamics (5), the governing dynamic equation for strategy adaptation in an $\mathcal{N}$-player CAID game can be expressed as:

$$\dot{z}_i(t) = \frac{1}{\mathcal{N}} \left(\sum_{j=1}^{\mathcal{N}} q_{ij}(z_j(t) - z_i(t)) \right). \tag{8}$$

Here, the given dynamic equation enables players to learn precisely from strategic differences, which deviates from real-world game scenarios. In [16], a finite time-based CAID dynamics model is incorporated, which integrates player learning with a discounted rate ($\gamma \in (0, 1)$), and is expressed as:

$$\dot{z}_i(t) = \frac{1}{\mathcal{N}} \left(\sum_{j=1}^{\mathcal{N}} q_{ij} \lfloor z_j(t) - z_i(t) \rceil^{\gamma} \right) \tag{9}$$

However, its convergence time depends upon the initial condition and takes more number of iterations to achieve consensus.

Therefore, this paper proposes a new CAID dynamics model based on the prescribed-time concept, incorporating player learning time that can be assigned a priori. The proposed CAID dynamic model is given as

$$\dot{z}_i(t) = \frac{1}{\mathcal{N}}\left(\alpha + \eta\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\sum_{j=1}^{\mathcal{N}} q_{ij}(z_j(t) - z_i(t)), \tag{10}$$

where $\alpha > 0$ is a constant, η is a positive constant, and function $\kappa(t)$ is defined in Lemma 1 in the subsequent section. The parameter η should be selected such that $\eta \geq \frac{1}{\lambda_{\min 2}(\mathbf{L}(q_{ij}))}$, where $\lambda_{\min 2}(\mathbf{L}(q_{ij}))$ is the second minimum eigenvalue of the Laplacian matrix $\mathbf{L}(q_{ij})$.

III. Convergence Analysis

A few lemmas are stated first that will be employed for the convergence proof of the proposed dynamic model (10).

Lemma 1: [19] (*Prescribed Time Convergence*) For a continuous system $\dot{x}(t) = g(x(t), t) \in \mathbb{R}^n$, $x(0) = x_0$, with origin being its equilibrium point, consider a function $\mathcal{W}(x(t), t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$, which is continuously differentiable. If $\exists$ a constant $c > 0$ such that the given conditions for $\mathcal{W}(x(t), t)$

$$\begin{cases} \mathcal{W}(0, t) = 0, \text{ and } \mathcal{W}(x(t), t) > 0 \text{ for } x(t) \in \mathbb{R}^n/\{0\} \\ \dot{\mathcal{W}}(x(t)) = -c\mathcal{W}(x(t)) - 2\frac{\dot{\kappa}(t)}{\kappa(t)}\mathcal{W}(x(t)) \text{ for } x(t) \in \mathbb{R}^n \end{cases} \tag{11}$$

holds on $[t_0, \infty)$ with

$$\kappa(t) = \begin{cases} \left(\frac{t_p}{t_p + t_0 - t}\right)^h, & t \in [t_0, t_0 + t_p) \\ 1 & t \in [t_0 + t_p, \infty) \end{cases} \tag{12}$$

and

$$\dot{\kappa}(t) = \begin{cases} \frac{h}{t_p}\kappa^{1+1/h}, & t \in [t_0, t_0 + t_p) \\ 0 & t \in [t_0 + t_p, \infty) \end{cases} \tag{13}$$

where h is a positive constant, t_0 represents the initial time, and t_p denotes the user-defined pre-assigned convergence time. Then, one can achieve

$$\begin{cases} \lim_{t \to (t_0 + t_p)^-} \mathcal{W}(x(t), t) = 0, \\ \mathcal{W}(x(t), t) = 0 \quad \forall\, t \geq t_0 + t_p. \end{cases} \tag{14}$$

Lemma 2: [12] The following properties are valid for an undirected connected graph $\mathcal{G}$:

1. $$\frac{1}{2}\sum_{i=1}^{\mathcal{N}} a_{ij}(x_j - x_i)^2 = \boldsymbol{x}^T\mathbf{L}\boldsymbol{x} \tag{15}$$

2. $$\min_{\boldsymbol{x} \neq 0, \mathbf{1}^T\boldsymbol{x} = 0} \frac{\boldsymbol{x}^T\mathbf{L}\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = \lambda_{\min 2}(\mathbf{L}) \tag{16}$$

where $\mathbf{L}$ represents the Laplacian matrix of $\mathcal{G}$, which is defined as $\mathbf{L} = [l_{ij}] \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$, and

$$l_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j \\ \sum_{k=1,\, k \neq i}^{\mathcal{N}} a_{ik} & \text{if } i = j \end{cases} \tag{17}$$

Lemma 3: [20] Given an undirected network with the condition $a_{ij} = a_{ji}$, then

$$\sum_{i=1}^{N}\delta_i\sum_{j=1}^{N} a_{ij}\mathcal{F}(\delta_i, \delta_j) = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(\delta_i - \delta_j)\mathcal{F}(\delta_i, \delta_j), \tag{18}$$

where $\mathcal{F}(\delta_i, \delta_j)$ is any function of δ_i and δ_j.

The prescribed time convergence analysis of the CAID problem is investigated in the following theorem.

Theorem 1: Suppose all players are fully connected. Then, the proposed strategy dynamics (10) solves the consensus iterative dilemma in a prescribed time t_p, that is pre-assigned by the user.

Proof: Define $\varpi = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} z_i(t)$ be the average strategy value of $\mathcal{N}$ players. Based on Equation (10), it can be inferred that ϖ remains constant because $\sum_{i=1}^{\mathcal{N}} \dot{z}_i(t) = 0$. Defining the error as $\varepsilon_i(t) = z_i(t) - \varpi$. Further,

$$\varepsilon_i(t) - \varepsilon_j(t) = z_i(t) - \varpi - z_j(t) + \varpi = z_i(t) - z_j(t) \tag{19}$$

Since ϖ is fixed, thus $\dot{\varepsilon}_i(t) = \dot{z}_i(t) - \dot{\varpi} = \dot{z}_i(t) - 0 = \dot{z}_i(t)$.

Now considering a Lyapunov candidate

$$\mathcal{V}(\varepsilon_i, t) = \frac{1}{2}\sum_{i=1}^{\mathcal{N}}\varepsilon_i^2(t) = \frac{1}{2}\boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t), \tag{20}$$

where $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \dots \varepsilon_{\mathcal{N}}]^T \in \mathbb{R}^{\mathcal{N}}$. The time derivative of $\mathcal{V}(\varepsilon_i, t)$ yields

$$\begin{aligned} \dot{\mathcal{V}}(\varepsilon_i, t) &= \sum_{i=1}^{\mathcal{N}}\varepsilon_i(t)\dot{\varepsilon}_i(t), \\ &= \sum_{i=1}^{\mathcal{N}}\varepsilon_i(t)\frac{1}{\mathcal{N}}\left(\alpha + \eta\frac{\dot{\kappa}(t)}{\kappa(t)}\right)\sum_{j=1}^{\mathcal{N}} q_{ij}(z_j(t) - z_i(t)). \end{aligned} \tag{21}$$

For brevity, the arguments of the function in the Lyapunov analysis are dropped in the latter part of the proof. Substituting the relation (19) in (21) gives

$$\begin{aligned} \dot{\mathcal{V}} &= \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\varepsilon_i\left(\alpha + \eta\frac{\dot{\kappa}}{\kappa}\right)\sum_{j=1}^{\mathcal{N}} q_{ij}(\varepsilon_j - \varepsilon_i) \\ &= \frac{\alpha}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\varepsilon_i\sum_{j=1}^{\mathcal{N}} q_{ij}(\varepsilon_j - \varepsilon_i) + \frac{\eta}{\mathcal{N}}\frac{\dot{\kappa}}{\kappa}\sum_{i=1}^{\mathcal{N}}\varepsilon_i\sum_{j=1}^{\mathcal{N}} q_{ij}(\varepsilon_j - \varepsilon_i) \end{aligned} \tag{22}$$

Using Lemma 3 yields

$$\begin{aligned} \dot{\mathcal{V}} &= \frac{\alpha}{2\mathcal{N}}\sum_{i,j=1}^{\mathcal{N}} q_{ij}(\varepsilon_i - \varepsilon_j)(\varepsilon_j - \varepsilon_i) + \frac{\eta}{2\mathcal{N}}\frac{\dot{\kappa}}{\kappa}\sum_{i,j=1}^{\mathcal{N}} q_{ij}(\varepsilon_i - \varepsilon_j) \\ &\quad \times (\varepsilon_j - \varepsilon_i) \\ &= -\frac{\alpha}{2\mathcal{N}}\sum_{i,j=1}^{\mathcal{N}} q_{ij}(\varepsilon_j - \varepsilon_i)(\varepsilon_j - \varepsilon_i) - \frac{\eta}{2\mathcal{N}}\frac{\dot{\kappa}}{\kappa}\sum_{i,j=1}^{\mathcal{N}} q_{ij}(\varepsilon_j - \varepsilon_i) \\ &\quad \times (\varepsilon_j - \varepsilon_i) \\ &= -\frac{\alpha}{2\mathcal{N}}\sum_{i,j=1}^{\mathcal{N}} q_{ij}(\varepsilon_j - \varepsilon_i)^2 - \frac{\eta}{2\mathcal{N}}\frac{\dot{\kappa}}{\kappa}\sum_{i,j=1}^{\mathcal{N}} q_{ij}(\varepsilon_j - \varepsilon_i)^2. \end{aligned} \tag{23}$$

Using both the properties of Lemma 2, one can rewrite (23) as

$$\begin{aligned}\dot{\mathcal{V}} &= -\frac{\alpha}{2\mathcal{N}}2\boldsymbol{\varepsilon}^T\mathbf{L}(q_{ij})\boldsymbol{\varepsilon} - \frac{\eta}{2\mathcal{N}}\frac{\dot{\kappa}}{\kappa}2\boldsymbol{\varepsilon}^T\mathbf{L}(q_{ij})\boldsymbol{\varepsilon} \\ &\le \frac{-\alpha}{\mathcal{N}}2\lambda_{\min 2}(\mathbf{L}(q_{ij}))\frac{\|\boldsymbol{\varepsilon}\|^2}{2} - \frac{\eta}{\mathcal{N}}\frac{\dot{\kappa}}{\kappa}2\lambda_{\min 2}(\mathbf{L}(q_{ij}))\frac{\|\boldsymbol{\varepsilon}\|^2}{2} \\ &\le -2\frac{\alpha}{\mathcal{N}}\lambda_{\min 2}(\mathbf{L}(q_{ij}))\frac{\|\boldsymbol{\varepsilon}\|^2}{2} - 2\frac{\eta}{\mathcal{N}}\frac{\dot{\kappa}}{\kappa}\lambda_{\min 2}(\mathbf{L}(q_{ij}))\frac{\|\boldsymbol{\varepsilon}\|^2}{2} \\ &= -2\frac{\alpha}{\mathcal{N}}\lambda_{\min 2}(\mathbf{L}(q_{ij}))\mathcal{V} - 2\frac{\eta}{\mathcal{N}}\lambda_{\min 2}(\mathbf{L}(q_{ij}))\frac{\dot{\kappa}}{\kappa}\mathcal{V}. \quad (24)\end{aligned}$$

Since from the design selection, $\eta \ge \frac{1}{\lambda_{\min 2}(\mathbf{L}(q_{ij}))}$. Therefore, $\eta\lambda_{\min 2}(\mathbf{L}(q_{ij})) \ge 1$. Also, from the multi-player system problem, $\mathcal{N} >> 1$. Consequently, $\frac{\eta}{\mathcal{N}}\lambda_{\min 2}(\mathbf{L}(q_{ij})) < 1$. Now, substituting this result in the Lyapunov analysis by taking the upper bound of inequality (24) gives

$$\begin{aligned}\dot{\mathcal{V}} &\le -2\frac{\alpha}{\mathcal{N}}\lambda_{\min 2}(\mathbf{L}(q_{ij}))\mathcal{V} - 2\frac{\dot{\kappa}}{\kappa}\mathcal{V} \\ &\le -c\mathcal{V} - 2\frac{\dot{\kappa}}{\kappa}\mathcal{V} \quad (25)\end{aligned}$$

where $c = 2\frac{\alpha}{\mathcal{N}}\lambda_{\min 2}(\mathbf{L}(q_{ij})) > 0$. The inequality (25) satisfies the prescribed time convergence condition (11) given in Lemma 1. Thus, as $\mathcal{V}(t)$ goes to zero for all time $t \ge t_p$, all the error variables $\varepsilon_i(t)$ for $i = 1$ to $\mathcal{N}$ will also go to zero for all time $t \ge t_p$. As a result, all the player's strategies $z_i(t)$ will achieve a consensus (i.e., $\varpi = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} z_i(t)$) within a prescribed time t_p, and interesting this prescribed time t_p is assigned by the user. Therefore, the convergence time can be selected beforehand, and it is independent of the initial player strategy. ■

Remark 1: Unlike the work in [16] with the strategy dynamics (9), the proposed strategy will not be affected by the chattering issue because there is no sign$(\cdot)$ function in dynamics (10).

IV. Numerical Analysis

This section demonstrates the validation of the proposed prescribed time strategy dynamic model and compares its convergence performance with the state-of-the-art finite-time convergent model [16]. Two well-known examples of game theory problems, i.e., prisoner's dilemma and snowdrift dilemma, are realized under both schemes to illustrate their effectiveness and convergence time. These classic dilemmas are used to model situations like corporate competition, international relations, diplomacy, and other relevant professions. These games showcase how individual and collective rationalities conflict or reason with each other. It effectively explains why seemingly rational actors might not cooperate, even when it benefits everyone [21].

Accordingly, continuous action iterative prisoner's dilemma (CAIPD) and continuous action iterative snowdrift dilemma (CAISD) are introduced as illustrative examples. The payoff matrices for these dilemmas are outlined as follows:

$$\text{CAIPD:} \quad \begin{bmatrix} g-p & -p \\ g & 0 \end{bmatrix} \quad (26)$$

$$\text{CAISD:} \quad \begin{bmatrix} g-p/2 & g-p \\ g & 0 \end{bmatrix} \quad (27)$$

where g stands for the benefit received by the individual, while p represents the cost borne by the cooperator. Further, $g > p$. The parameter values of the playoff matrices and both the schemes are written in Table I.

TABLE I: Parameter values

System/Scheme	Parameters
CAIPD, CAISD	$g = 5$, $p = 1$, $\mathcal{N} = 42$, $z_i(0) \in (0, 1)$
Finite time [16]	$\beta = 0.5$, $\epsilon = 0.5$, $\alpha = 0.5$
Proposed prescribed time	$h = 1.45$, $\alpha = \mu = 0.5$, $\eta = 2$, $\wp = 1$ $t_p = 2$s for CAIPD, $t_p = 1$s for CAISD

The game is set up with $\mathcal{N} = 42$ agents, and each agent starts with a random strategy between 0 and 1. The simulation is conducted to see if all the agents can eventually agree on the same strategy based on the rules of continuous action iterated dilemma.

A. Numerical Results of CAIPD

The simulation results of the proposed prescribed time and comparative finite time [16] for the CAIPD game in the fully connected network are depicted in Fig. 1. The convergence response of the prescribed time strategy (top subplot of Fig. 1) illustrates faster convergence to the consensus with fewer iterations compared to the state-of-the-art method. Further, the convergence time is achieved at the exact assigned time, i.e., $t_p = 2$s.

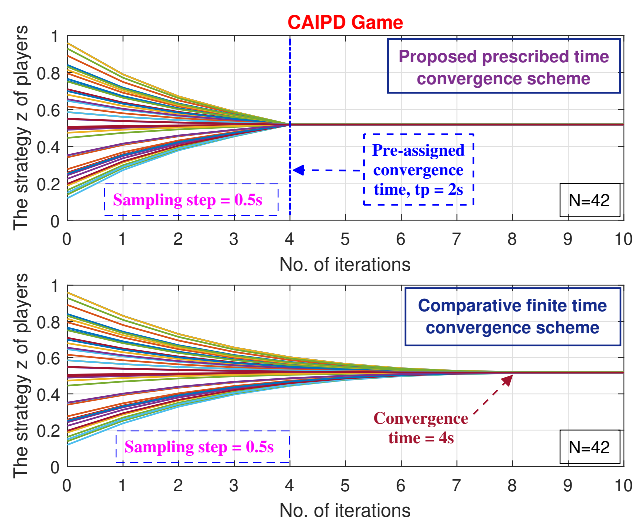

Fig. 1: Convergence analysis for CAIPD under both schemes.

B. Numerical Results of CAISD

In this example, the pre-assigned time is selected as $t_p = 1$s. The simulation results for the CAISD game within a fully connected network are presented in Fig. 2. The players' strategy in the prescribed time scheme demonstrates quicker convergence to a common value at the allotted time $t_p = 1$s, as seen in the top subplot of Fig. 2. Moreover, the number of iterations is also less in the proposed strategy compared to the

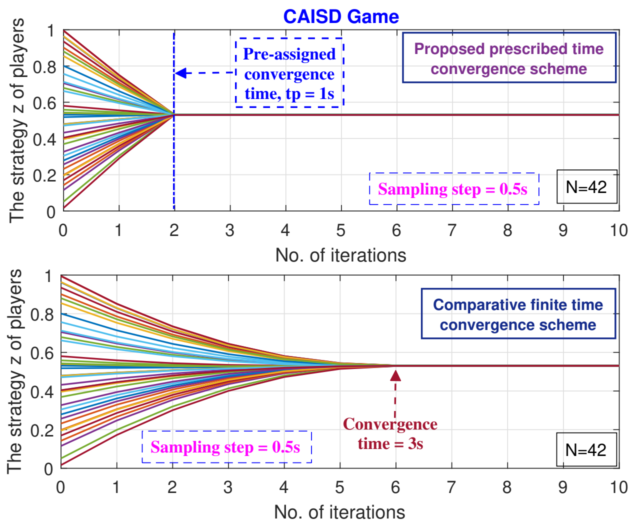

Fig. 2: Convergence analysis for CAISD under both schemes.

finite time scheme. The finite time strategy takes 6 iterations and 3s to converge.

Table II presents the performance comparison based on convergence time (T_c) and number of iterations. The table clearly highlights the superior performance of the proposed prescribed time method over the finite time scheme for both games.

TABLE II: Performance of CAID games under both schemes.

Games →	**CAIPD**		**CAISD**	
Schemes ↓	No. of iterations	T_c	No. of iterations	T_c
Finite time [16]	8	4s	6	3s
Prescribed time	4	2s	2	1s

V. Conclusion

This paper analyzes the convergence of CAID, allowing players to explore a much wider range of strategic options. A novel dynamic strategy is adapted using the prescribed time convergence method, achieving two key milestones: convergence of strategy consensus is independent of the initial strategies, and convergence time is tunable by the user in advance. These features are achieved by adapting the player's learning rate with a predefined prescribed function. The prescribed time convergence is affirmed by Lyapunov stability analysis. Further, numerical analysis is carried out to compare the proposed method with the state-of-the-art finite-time method for CAIPD and CAISD games. These comparisons convincingly demonstrate that the proposed approach achieves faster convergence and requires fewer iterations.

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References

[1] W. Cheng, Y. Zou, J. Xu, and W. Liu, "Dynamic games for social model training service market via federated learning approach," *IEEE Trans. Computational Social Systems*, vol. 9, no. 1, pp. 64–75, 2021.

[2] G. Zhao, Y. Wang, and H. Li, "A matrix approach to the modeling and analysis of networked evolutionary games with time delays," *IEEE/CAA J. Autom. Sinica*, vol. 5, no. 4, pp. 818–826, 2018.

[3] P. Zhu, X. Wang, D. Jia, Y. Guo, S. Li, and C. Chu, "Investigating the co-evolution of node reputation and edge-strategy in prisoner's dilemma game," *Applied Math. & Computation*, vol. 386, p. 125474, 2020.

[4] H. Ito and J. Tanimoto, "Scaling the phase-planes of social dilemma strengths shows game-class changes in the five rules governing the evolution of cooperation," *Royal Society open science*, vol. 5, no. 10, p. 181085, 2018.

[5] E. Mazumdar, L. J. Ratliff, and S. S. Sastry, "On gradient-based learning in continuous games," *SIAM Journal on Mathematics of Data Science*, vol. 2, no. 1, pp. 103–131, 2020.

[6] Z. Wang, H. Li, X. Jin, D. Yu, K. H. Cheong, and X. Li, "Complex continuous action iterated dilemma with incremental dynamic model," *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 54, no. 4, pp. 2309–2319, 2024.

[7] J. Zhang, J. Lou, J. Qiu, and J. Lu, "Dynamics and convergence of hyper-networked evolutionary games with time delay in strategies," *Information Sciences*, vol. 563, pp. 166–182, 2021.

[8] Y. Zheng, C. Li, and J.-e. Feng, "Modeling and dynamics of networked evolutionary game with switched time delay," *IEEE Trans. Control of Network Systems*, vol. 8, no. 4, pp. 1778–1787, 2021.

[9] T. Mai, H. Yao, N. Zhang, L. Xu, M. Guizani, and S. Guo, "Cloud mining pool aided blockchain-enabled internet of things: An evolutionary game approach," *IEEE Trans. Cloud Computing*, vol. 11, no. 1, pp. 692–703, 2021.

[10] B. Ranjbar-Sahraei, H. Bou Ammar, D. Bloembergen, K. Tuyls, and G. Weiss, "Evolution of cooperation in arbitrary complex networks," in *Proc. Int. Conf. Autonomous Agents and Multi-Agent Systems*, 2014, pp. 677–684.

[11] P. Ramazi and M. Cao, "Global convergence for replicator dynamics of repeated snowdrift games," *IEEE Trans. Automatic Control*, vol. 66, no. 1, pp. 291–298, 2020.

[12] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[13] X. Jin, Z. Wang, D. Yu, and X. Li, "The convergence analysis of evolutionary dynamics for continuous action iterated dilemma in information loss networks," *IEEE Trans. Computational Social Systems*, 2023.

[14] P. Zhu, J. Sun, D. Yu, C. Liu, Y. Zhou, and Z. Wang, "Continuous action iterated dilemma under double-layer network with unknown nonlinear dynamics and its convergence analysis," *Nonlinear Dynamics*, vol. 111, no. 23, pp. 21 611–21 625, 2023.

[15] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.

[16] Z. Wang, X. Jin, T. Zhang, and D. Yu, "A finite-time convergent analysis of continuous action iterated dilemma," *IEEE/CAA J. Autom. Sinica*, vol. 11, no. 2, pp. 563–565, 2024.

[17] Y. Song, Y. Wang, J. Holloway, and M. Krstic, "Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time," *Automatica*, vol. 83, pp. 243–251, 2017.

[18] Y. Song, H. Ye, and F. L. Lewis, "Prescribed-time control and its latest developments," *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 53, no. 7, pp. 4102–4116, 2023.

[19] Y. Wang, Y. Song, D. J. Hill, and M. Krstic, "Prescribed-time consensus and containment control of networked multiagent systems," *IEEE Trans. Cybernetics*, vol. 49, no. 4, pp. 1138–1147, 2019.

[20] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Automatic Control*, vol. 55, no. 4, pp. 950–955, 2010.

[21] R. Kümmerli, C. Colliard, N. Fiechter, B. Petitpierre, F. Russier, and L. Keller, "Human cooperation in social dilemmas: comparing the snowdrift game with the prisoner's dilemma," *Proceedings of the Royal Society B: Biological Sciences*, vol. 274, no. 1628, pp. 2965–2970, 2007.